Methods of Mathematical Physics

Preface:

I thank Allah Almighty who gave me a chance to serve the students in the field of Mathematics. I also thank the Holy Prophet Hazrat Muhammad (PBUH), who is forever a torch of guidance for humanity, without his goodwill nothing can be achieved. I am very grateful to my friends who encouraged me during the preparation of these notes. Further, I am thankful to my respected teacher **Prof. Dr.Azizullah Awan** for his kind support in this regard.

I feel myself lucky enough that my notes are popular all over Pakistan. Suggestions, comments, and indications of errors for the improvement will be highly appreciated.

I pray that my efforts will help the students in achieving their future objectives.

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Notes

M.Sc Part II (PU)

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Section I:

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CHAP 1:

Power Series Solution.

Differential Equation.

An equation involving one dependent variable and its derivatives with respect to one or more independent variables, is called differential equation.

For example, (i) $\frac{dy}{dx} + y\cos x = \sin x$

(ii) $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$

(iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

are differential equations.

Ordinary Differential Equation (ODE).

A differential equation, in which ordinary derivatives of the dependent variable w.r.t. a single independent variable occur, is called an ODE.

For example (i) $\frac{dy}{dx} + y\cos x = \sin x$

(ii) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$

are ODE.

Partial Differential Equation (PDE).

A differential equation involving partial derivatives of the dependent variable w.r.t. more than one independent variable is called a PDE.

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For example, (i) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx$

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

are PDE's.

Order of DE.

The order of DE is the order of the highest derivatives that occurs in the equation.

Degree of DE.

The degree of differential equation is the greatest exponent of the higest order derivative that appears in the equation.

Solution of DE.

A solution of a DE is a relation b/w the variables, not containing derivatives, such that this relation and the derivatives obtained from it satisfy the given DE identically.

For example the equation

$\frac{dy}{dx} = -\lambda y$ has a solution $y = ce^{-\lambda x}$.

General Solution.

A solution of DE which contains as many arbitrary constants as the order of the equation is called general solution of the DE.

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Particular Solution.

A solution Obtained from the general solution by giving particular values to the constants is called a particular solution of DE.

Linear DE. 2010,

An ordinary DE $F(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}) = 0$

is said to be linear if F is a linear function of the variables $x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$.

A linear diff. equation must satisfy the following three conditions.

(i) The dependent variable y and its derivatives are all of degree one.

(ii) No products of y or any of its derivatives appear.

(iii) No transcedental function of y and/or its derivatives occur.

Homogeneous DE.

A DE which has a trivial solution is called homogeneous DE.

For example, $x + y = 0$.

Non homogeneous DE.

A DE which has no trivial solution is called non homogeneous DE.

For example, $x + y + 5 = 0$

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$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + f(x,y)u = g(x,y).$$

① The PDE is parabolic if $b^2 - 4ac = 0$

For example $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$

② The PDE is hyperbolic if $b^2 - 4ac > 0$

For example $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$

③ The PDE is elliptic if $b^2 - 4ac < 0$

For example $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Power series.

An infinite series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$$

is called power series centered at a.

Convergence of Power series.

A power series $\sum_{n=0}^{\infty} C_n (x-a)^n$ is convergent at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges

i.e. $\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} C_n (x-a)^n$ exists

If the limit does not exist at x, the series is said to be divergent.

Radius of Convergence.

Every power series has a radius of Convergence R.

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① If $R>0$, then a power series $\sum_{n=0}^{\infty} C_n(x-a)^n$ converges for $|x-a|<R$ and diverges for $|x-a|>R$.

② If the series converges only at its centre a, then $R=0$

③ If series converges for all x, then $R=\infty$.

Interval of Convergence.

The interval of convergence is the set of all real numbers x for which the series converges

Note:

① If $y=\sum_{n=0}^{\infty} C_n x^n$ is a power series in x, then its 1st two derivatives are

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}.$$

② $$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Identity property:

If $\sum_{n=0}^{\infty} C_n(x-a)^n = 0$, $R>0$ for all numbers x in the interval of convergence, then $C_n = 0 \ \forall n$.

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Analytic Function.

A function f is said to be analytic at a point "a" if it can be represented by a power series in "x−a" with a +ive radius of convergence.

e.g. e^x, Sinx, Cosx etc.

Addition of two Power Series.

In order to add the two power series, it is necessary that both summation indices starts with the same number and that the powers of x in each series should be same.

Power Series Solution.

The linear 2nd order DE is given as

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

Dividing by the leading coefficent $a_2(x)$, we get

$$y'' + \frac{a_1(x)}{a_2(x)}\,y' + \frac{a_0(x)}{a_2(x)}\,y = 0$$

and the standard form is

$$y'' + P(x)\,y' + Q(x)\,y = 0.$$

where $P(x) = \frac{a_1(x)}{a_2(x)}$ and $Q(x) = \frac{a_0(x)}{a_2(x)}$

Ordinary Point. 2011, 2013, 2015,

A point $x = x_0$ is said to be an ordinary point of the differential equation

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$$y'' + P(x)y' + Q(x)y = 0$$

if both $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

For example $x=0$ is an ordinary point of DE $y'' + e^x y' + (\sin x)y = 0$.

Singular Point. 2013, 2015,

A point $x=x_0$ is said to be a singular point of DE $y'' + P(x)y' + Q(x)y = 0$ if $P(x)$ or $Q(x)$ are not analytic at $x=x_0$.

For example $x=0$ is a singular point of DE $y'' + e^x y' + (\ln x)y = 0$.

Note:

The singular points need not to be real.

For example the equation

$$(x^2+1)y'' + xy' - y = 0$$

$$\Rightarrow \quad y'' + \frac{x}{x^2+1}y' - \frac{1}{x^2+1}y = 0$$

has singular points at $x = \pm i$.

Theorem.

If $x=x_0$ is an ordinary point of the DE $y'' + P(x)y' + Q(x)y = 0$, we can always find two linearly independent solutions in the form of a power series centered at $x=x_0$

i.e. $y = \sum_{n=0}^{\infty} C_n (x-a)^n$.

A series solution converges at least on some

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interval defind as $|x-x_0|<R$, where R is the distance from x_0 to the closest singular point.

(The distance R is the minimum value or lower bound for the radius of convergence.)

Example.

Solve $y''+xy=0$

Slow

$$y''+2xy=0$$

Solution.

Given that $y''+xy=0 \longrightarrow (1)$

Clearly this DE has no singular point which guarantees two power series solutions centred at 0, and those solutions converge for $|x|<\infty$.

Subsituting $y=\sum_{n=0}^{\infty} C_n x^n \longrightarrow (2)$

in (1), we get,

$$y''=\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}$$

$$(1)\Rightarrow \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}+x\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}+\sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\Rightarrow 2C_2+\sum_{n=3}^{\infty} n(n-1)C_n x^{n-2}+\sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\Rightarrow 2C_2+\sum_{K+2=3}^{\infty} (K+2)(K+1)C_{K+2} x^K+\sum_{K-1=0}^{\infty} C_{K-1} x^K = 0$$

$$\Rightarrow 2C_2+\sum_{K=1}^{\infty} (K+2)(K+1)C_{K+2} x^K+\sum_{K=1}^{\infty} C_{K-1} x^K = 0$$

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$$\Rightarrow 2C_2 + \sum_{K=1}^{\infty} \left[(K+2)(K+1)C_{K+2} + C_{K-1} \right] x^K = 0$$

using identity properly, we get,

$2C_2 = 0$; $(K+2)(K+1)C_{K+2} + C_{K-1} = 0$

$\Rightarrow C_2 = 0$; $C_{K+2} = -\frac{C_{K-1}}{(K+2)(K+1)}$, $K = 1, 2, 3, \cdots$

$K=1 \Rightarrow C_3 = -\frac{C_0}{3 \cdot 2}$; $K=3 \Rightarrow C_5 = \frac{-C_2}{5 \cdot 4} = 0$ $\because C_2 = 0$

$K=2 \Rightarrow C_4 = \frac{-C_1}{4 \cdot 3}$; $K=4 \Rightarrow C_6 = \frac{-C_3}{6 \cdot 5} = \frac{C_0}{6 \cdot 5 \cdot 3 \cdot 2}$

$K=5 \Rightarrow C_7 = \frac{-C_4}{7 \cdot 6} = \frac{C_1}{7 \cdot 6 \cdot 4 \cdot 3}$

$K=6 \Rightarrow C_8 = \frac{-C_5}{8 \cdot 7} = 0$ $\because C_5 = 0$.

$K=7 \Rightarrow C_9 = \frac{-C_6}{9 \cdot 8} = -\frac{C_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$

$K=8 \Rightarrow C_{10} = \frac{-C_7}{10 \cdot 9} = -\frac{C_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$

$K=9 \Rightarrow C_{11} = \frac{-C_8}{11 \cdot 10} = 0$ $\because C_8 = 0$.

and so on.

Putting these values of C_i's in (2), we have,

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$\Rightarrow y = C_0 + C_1 x - \frac{C_0}{3 \cdot 2} x^3 - \frac{C_1}{4 \cdot 3} x^4 + \frac{C_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{C_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7$$

$$- \frac{C_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} x^9 - \frac{C_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3} x^{10} + \cdots$$

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$$= C_0\left(1 - \frac{1}{3\cdot 2}x^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2}x^6 - \frac{1}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}x^9 + \cdots\right)$$

$$+ C_1\left(x - \frac{1}{4\cdot 3}x^4 + \frac{1}{7\cdot 6\cdot 4\cdot 3}x^7 - \frac{1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}x^{10} + \cdots\right)$$

$$= C_0\left(1 + \sum_{K=1}^{\infty} \frac{(-1)^K}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9\cdots(3K-1)(3K)}x^{3K}\right)$$

$$+ C_1\left(x + \sum_{K=1}^{\infty} \frac{(-1)^K}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10\cdots(3K)(3K+1)}x^{3K+1}\right)$$

$$\Rightarrow y = C_0 y_1(x) + C_1 y_2(x)$$

Where $y_1(x) = 1 + \sum_{K=1}^{\infty} \frac{(-1)^K}{2\cdot 3\cdot 5\cdot 6\cdots(3K-1)(3K)}x^{3K}$

and $y_2(x) = x + \sum_{K=1}^{\infty} \frac{(-1)^K}{3\cdot 4\cdot 6\cdot 7\cdots(3K)(3K+1)}x^{3K+1}$

are two linearly independent series solutions of (1) and these solutions converge for $|x| < \infty$.

~~$y'' + 2xy' + y = 0$~~

Question 1. $y'' - 2xy' + y = 0$

Solution.

Given that $y'' - 2xy' + y = 0 \longrightarrow (1)$

Clearly this DE has no singular point which guarantees two power series solutions centred at 0 and those solutions converge for $|x| < \infty$.

Subsituting $y = \sum_{n=0}^{\infty} C_n x^n \longrightarrow (2)$

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in (1), we get,

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

and $$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2x \sum_{n=1}^{\infty} n C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2 \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow 2C_2 + \sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} - 2 \sum_{n=1}^{\infty} n C_n x^n + C_0 + \sum_{n=1}^{\infty} C_n x^n = 0$$

$$\Rightarrow C_0 + 2C_2 + \sum_{K=1}^{\infty} (K+2)(K+1) C_{K+2} x^K - 2 \sum_{K=1}^{\infty} K C_K x^K + \sum_{K=1}^{\infty} C_K x^K = 0$$

$$\Rightarrow C_0 + 2C_2 + \sum_{K=1}^{\infty} \left[(K+2)(K+1) C_{K+2} - 2K C_K + C_K \right] x^K = 0$$

using identity properly, we get,

$$2C_2 + C_0 = 0 \; ; \; (K+2)(K+1) C_{K+2} - 2K C_K + C_K = 0$$

$$\Rightarrow C_2 = \frac{-1}{2} C_0 \; ; \quad C_{K+2} = \frac{2K-1}{(K+2)(K+1)} C_K \; ; \; K = 1, 2, 3, \cdots$$

$$K=1 \Rightarrow C_3 = \frac{1}{3 \cdot 2} C_1 = \frac{1}{3!} C_1$$

$$K=2 \Rightarrow C_4 = \frac{3}{4 \cdot 3} C_2 = \frac{-3}{4 \cdot 3 \cdot 2} C_0 = \frac{-3}{4!} C_0$$

$$K=3 \Rightarrow C_5 = \frac{5}{5 \cdot 4} C_3 = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} C_1 = \frac{5}{5!} C_1$$

$$K=4 \Rightarrow C_6 = \frac{7}{6 \cdot 5} C_4 = \frac{-7 \cdot 3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} C_0 = \frac{1}{6!} C_0$$

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$K=5 \Rightarrow C_7 = \frac{9}{7\cdot6} C_5 = \frac{9\cdot5}{7\cdot6\cdot5\cdot4\cdot3\cdot2} C_1 = \frac{9\cdot5}{7!} C_1$

$K=6 \Rightarrow C_8 = \frac{11}{8\cdot7} C_6 = \frac{-11\cdot7\cdot3}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} C_0 = \frac{-11\cdot7\cdot3}{8!} C_0$

$K=7 \Rightarrow C_9 = \frac{13}{9\cdot8} C_7 = \frac{13\cdot9\cdot5}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} C_1 = \frac{13\cdot9\cdot5}{9!} C_1$

And so on.

Putting these values of C_i's in (2), we get,

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$\Rightarrow y = C_0 + C_1 x - \frac{1}{2!} C_0 x^2 + \frac{1}{3!} C_1 x^3 - \frac{3}{4!} C_0 x^4 + \frac{5}{5!} C_1 x^5$$

$$- \frac{7\cdot3}{6!} C_0 x^6 + \frac{9\cdot5}{7!} C_1 x^7 - \frac{11\cdot7\cdot3}{8!} C_0 x^8 + \frac{13\cdot9\cdot5}{9!} C_1 x^9$$

$$- \cdots$$

$$\Rightarrow y = C_0 \left(1 - \frac{1}{2!} x^2 - \frac{3}{4!} x^4 - \frac{7\cdot3}{6!} x^6 - \frac{11\cdot7\cdot3}{8!} x^8 - \cdots\right)$$

$$+ C_1 \left(x + \frac{1}{3!} x^3 + \frac{5}{5!} x^5 + \frac{9\cdot5}{7!} x^7 + \frac{13\cdot9\cdot5}{9!} x^9 + \cdots\right)$$

$$\Rightarrow y = C_0 \left(1 - \sum_{K=1}^{\infty} \frac{3\cdot7\cdot11\cdots(4K-1)}{2K!} x^{2K}\right) + C_1 \left(x + \sum_{K=1}^{\infty} \frac{5\cdot9\cdot13\cdots(4K+1)}{(2K+1)!} x^{2K+1}\right)$$

$$\Rightarrow y = C_0 y_1(x) + C_1 y_2(x)$$

where $y_1(x) = 1 - \sum_{K=1}^{\infty} \frac{3\cdot7\cdot11\cdots(4K-1)}{2K!} x^{2K}$

and $y_2(x) = x + \sum_{K=1}^{\infty} \frac{5\cdot9\cdot13\cdots(4K+1)}{(2K+1)!} x^{2K+1}$

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are L.I. series solutions of (1) and these solutions converge for $|x|<\infty$.

Q2: $y'' - xy = 0$ 2012,

Answer: $y_1(x) = 1 + \sum_{K=1}^{\infty} \frac{1}{2\cdot3\cdot5\cdot6\cdots(3K-1)(3K)} x^{3K}$

and $y_2(x) = x + \sum_{K=1}^{\infty} \frac{1}{3\cdot4\cdot6\cdot7\cdots(3K)(3K+1)} x^{3K+1}$

Q3: $y'' + x^2 y = 0$

Answer. $y_1(x) = 1 + \sum_{K=1}^{\infty} \frac{(-1)^K}{3\cdot4\cdot7\cdot8\cdots(4K-1)(4K)} x^{4K}$

and $y_2(x) = x + \sum_{K=1}^{\infty} \frac{(-1)^K}{4\cdot5\cdot8\cdot9\cdots(4K)(4K+1)} x^{4K+1}$

Q4: $y'' - xy' + 2y = 0$

Answer. $y_1(x) = 1 - \frac{1}{2} x^2$

and $y_2(x) = x - \sum_{K=1}^{\infty} \frac{1\cdot3\cdot5\cdots(2K+1)}{(2K+1)!} x^{2K+1}$

Q5: $y'' + x^2 y' + xy = 0$

Answer. $y_1(x) = 1 + \sum_{K=1}^{\infty} \frac{(-1)^K 1\cdot4\cdot7\cdots(3K-2)}{2\cdot3\cdot5\cdot6\cdots(3K-1)(3K)} x^{3K}$

and

$y_2(x) = x + \sum_{K=1}^{\infty} \frac{(-1)^K 2\cdot5\cdot8\cdots(3K-1)}{3\cdot4\cdot6\cdot7\cdot9\cdot10\cdots(3K)(3K+1)} x^{3K+1}$

Q6: $y'' + 2xy' + 2y = 0$

Answer. $y_1(x) = 1 + \sum_{K=1}^{\infty} \frac{(-1)^K 2^K}{2\cdot4\cdot6\cdots2K} x^{2K}$

and $y_2(x) = x + \sum_{K=1}^{\infty} \frac{(-1)^K 2^K}{3\cdot5\cdot7\cdots(2K+1)}$

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Example.

Solve $(x^2+1)y'' + xy' - y = 0$.

Solution.

Given that $(x^2+1)y'' + xy' - y = 0$ ——— (1)

Clearly $x = \pm i$ are two singular points of this DE and it guarantees two power series solutions at ordinary point $x=0$. These two linearly independent series solutions will converge at least for $|x-0| < 1$, i.e. $|x| < 1$, where 1 is the distance in the complex plane from 0 to $\pm i$.

Subsituting $y = \sum_{n=0}^{\infty} c_n x^n$ ——— (2)

in (1), we get,

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n + 2c_2 + 3\cdot 2 c_3 x + \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}$$

$$+ c_1 x + \sum_{n=2}^{\infty} n c_n x^n - c_0 - c_1 x - \sum_{n=2}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2c_2 + 3\cdot 2 c_3 x + c_1 x - c_0 - c_1 x + \sum_{n=2}^{\infty} n(n-1)c_n x^n$$

$$+ \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n c_n x^n - \sum_{n=2}^{\infty} c_n x^n = 0$$

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$$\Rightarrow (2C_2 - C_0) + 3\cdot 2\, C_3 x + \sum_{K=2}^{\infty} K(K-1) C_K x^K + \sum_{K=2}^{\infty} (K+2)(K+1) C_{K+2} x^K$$

$$+ \sum_{K=2}^{\infty} K C_K x^K - \sum_{K=2}^{\infty} C_K x^K = 0$$

$$\Rightarrow (2C_2 - C_0) + 3\cdot 2 C_3 x + \sum_{K=2}^{\infty} \Big[K(K-1) C_K + (K+2)(K+1) C_{K+2}$$

$$+ K C_K - C_K \Big] x^K = 0$$

Using Identity Properly, we get,

$$2C_2 - C_0 = 0 \; ; \; 3\cdot 2 C_3 = 0$$

$$; \; K(K-1) C_K + (K+2)(K+1) C_{K+2} + K C_K - C_K = 0$$

$$\Rightarrow C_2 = \frac{1}{2} C_0 \; ; \; C_3 = 0 ;$$

$$\Big[K(K-1) + K - 1 \Big] C_K + (K+2)(K+1) C_{K+2} = 0$$

$$\Rightarrow (K-1)(K+1) C_K + (K+2)(K+1) C_{K+2} = 0$$

$$\Rightarrow C_{K+2} = \frac{-(K-1) C_K}{K+2} = \frac{1-K}{K+2} C_K$$

where $K = 2, 3, 4, \cdots$

$$K=2 \Rightarrow C_4 = \frac{-C_2}{4} = \frac{-1}{4\cdot 2} C_0 = \frac{-1}{2^2 (2\cdot 1)} C_0$$

$$K=3 \Rightarrow C_5 = \frac{-2}{5} C_3 = 0 \quad \because C_3 = 0$$

$$K=4 \Rightarrow C_6 = \frac{-3}{6} C_4 = \frac{3}{6\cdot 4\cdot 2} C_0 = \frac{3}{2^3 (3\cdot 2\cdot 1)} C_0$$

$$K=5 \Rightarrow C_7 = \frac{-4}{7} C_5 = 0 \quad \because C_5 = 0$$

$$K=6 \Rightarrow C_8 = \frac{-5}{8} C_6 = \frac{-5\cdot 3}{8\cdot 6\cdot 4\cdot 2} C_0 = \frac{-5\cdot 3\, C_0}{2^4 (4\cdot 3\cdot 2\cdot 1)}$$

and so on.

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Putting these values of C_i's in (2), we get

$$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \cdots$$

$$\Rightarrow y = C_0 + C_1 x + \frac{1}{2} C_0 x^2 - \frac{1}{2^2 2!} C_0 x^4 + \frac{3}{2^3 3!} C_0 x^6 - \frac{5 \cdot 3}{2^4 4!} C_0 x^8 + \cdots$$

$$\Rightarrow y = C_0 \left(1 + \frac{1}{2} x^2 - \frac{1}{2^2 2!} x^4 + \frac{3 \cdot 1}{2^3 3!} x^6 - \frac{5 \cdot 3 \cdot 1}{2^4 4!} x^8 + \cdots\right) + C_1 x$$

$$\Rightarrow y = C_0 \left(1 + \frac{1}{2} x^2 + \sum_{K=2}^{\infty} \frac{(-1)^{K-1} \, 1 \cdot 3 \cdot 5 \cdots (2K-3)}{2^K K!} x^{2K}\right) + C_1 x.$$

$$\Rightarrow y = C_0 y_1(x) + C_1 y_2(x)$$

where

$$y_1(x) = 1 + \frac{1}{2} x^2 + \sum_{K=2}^{\infty} \frac{(-1)^{K-1} \, 1 \cdot 3 \cdot 5 \cdots (2K-3)}{2^K K!} x^{2K}$$

and $y_2(x) = x$ are two l.I. solutions of (1) and these solutions converge at least for $|x| < 1$.

Question 1:

solve $(x^2 - 1) y'' + xy' - y = 0$

Solution.

Given that $(x^2 - 1) y'' + xy' - y = 0 \rightarrow (1)$

Clearly this DE has singular points at $x = \pm 1$ which gaurantees two power series solutions at ordinary

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point $x=0$. These two linearly independent series solutions converge at least for $|x|<1$, where 1 is the minimum distance from 0 to ± 1.

Substituting $y=\sum_{n=0}^{\infty} c_n x^n$ ——— (2) in (1), we have

$$(x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - 2c_2 - 3\cdot 2 c_3 x - \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + c_1 x + \sum_{n=2}^{\infty} n c_n x^n - c_0 - c_1 x - \sum_{n=2}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n c_n x^n - \sum_{n=2}^{\infty} c_n x^n - 2c_2 - 3\cdot 2 c_3 x - c_0 = 0$$

$$\Rightarrow (-2c_2 - c_0) - 3\cdot 2 c_3 x + \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=2}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=2}^{\infty} k c_k x^k - \sum_{k=2}^{\infty} c_k x^k = 0$$

$$\Rightarrow (-2c_2 - c_0) - 6c_3 x + \sum_{k=2}^{\infty} \left[k(k-1)c_k - (k+2)(k+1)c_{k+2} + k c_k - c_k \right] x^k = 0$$

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Using Identity properly, we get,

$-2C_2 - C_0 = 0 \quad ; 6C_3 = 0$

$K(K-1)C_K - (K+2)(K+1)C_{K+2} + KC_K - C_K = 0$

$\Rightarrow C_2 = \frac{-1}{2} C_0 \quad ; \quad C_3 = 0$

$[K(K-1) + (K-1)]C_K - (K+2)(K+1)C_{K+2} = 0$

$\Rightarrow (K-1)(K+1)C_K = (K+2)(K+1)C_{K+2}$

$\Rightarrow C_{K+2} = \frac{K-1}{K+2} C_K \quad ; K = 2, 3, \cdots$

$K = 2 \Rightarrow C_4 = \frac{1}{4} C_2 = \frac{-1}{4 \cdot 2} C_0$

$K = 3 \Rightarrow C_5 = \frac{2}{5} C_3 = 0 \quad \because C_3 = 0$

$K = 4 \Rightarrow C_6 = \frac{3}{6} C_4 = \frac{-3 \cdot 1}{6 \cdot 4 \cdot 2} C_0$

$K = 5 \Rightarrow C_7 = \frac{4}{7} C_5 = 0 \quad \because C_5 = 0$

$K = 6 \Rightarrow C_8 = \frac{5}{8} C_6 = \frac{-5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} C_0$

$K = 7 \Rightarrow C_9 = 0$

$K = 8 \Rightarrow C_{10} = \frac{7}{10} C_8 = \frac{-7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} C_0$

and so on.

Putting these values of C_i's in (2), we get,

$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$

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$$\Rightarrow y = c_0 + c_1 x - \frac{1}{2} c_0 x^2 - \frac{1}{4\cdot 2} c_0 x^4 - \frac{3\cdot 1}{6\cdot 4\cdot 2} x^6 - \frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2} x^8 - \frac{7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} x^{10} - \cdots$$

$$\Rightarrow y = c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{4\cdot 2}x^4 - \frac{3\cdot 1}{6\cdot 4\cdot 2}x^6 - \frac{5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}x^8 - \frac{7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2}x^{10} - \cdots\right) + c_1 x$$

$$\Rightarrow y = c_0\left(1 - \frac{1}{2}x^2 - \sum_{k=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7 \cdots (2k-3)}{2\cdot 4\cdot 6 \cdots (2k-2)} x^{2k}\right) + c_1 x$$

$$\Rightarrow y = c_0 y_1(x) + c_1 y_2(x)$$

where $y_1(x) = 1 - \frac{1}{2}x^2 - \sum_{k=2}^{\infty} \frac{1\cdot 3\cdot 5\cdot 7 \cdots (2k-3)}{2\cdot 4\cdot 6 \cdots (2k-2)} x^{2k}$

and $y_2(x) = x$

are two L.I. solutions of ① and these solutions converge at least for $|x| < 1$.

Solve.

② $(x-1)y'' + y' = 0$ ③ $(x+2)y'' + xy' - y = 0$

④ $y'' - (x+1)y' - y = 0$ ⑤ $(x^2+1)y'' - 6y = 0$

⑥ $(x^2+2)y'' + 3xy' - y = 0$

Example.

Solve, $y'' - (1+x)y = 0$

Solution.

Given that $y'' - (1+x)y = 0$ ——— ①

Clearly this DE has no singular point which guarantees two power series solution centred at 0 and those solutions converge for $|x| < \infty$.

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Subsituting $y = \sum_{n=0}^{\infty} C_n x^n$ ——— (2)

in (1), we get,

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - (1+x) \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=0}^{\infty} C_n x^n - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\Rightarrow 2C_2 + \sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} - C_0 - \sum_{n=1}^{\infty} C_n x^n - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

$$\Rightarrow 2C_2 - C_0 + \sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^k - \sum_{k=1}^{\infty} C_k x^k - \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$$\Rightarrow (2C_2 - C_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1) C_{k+2} - C_k - C_{k-1} \right] x^k = 0$$

Using identity properly, we get,

$2C_2 - C_0 = 0$; $(k+2)(k+1) C_{k+2} - C_k - C_{k-1} = 0$

$\Rightarrow C_2 = \frac{1}{2} C_0$; $(k+2)(k+1) C_{k+2} = C_k + C_{k-1}$

$$\Rightarrow C_{k+2} = \frac{C_k + C_{k-1}}{(k+2)(k+1)} \quad ; k = 1, 2, 3, \cdots$$

This shows that the coefficients $C_3, C_4, \cdots$ are expressed in terms of both C_1 and C_0 and moreover the algebra required to do this becomes a little messy.

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Case 1.

we Choose $C_0 \neq 0$, $C_1 = 0$.

$$C_2 = \frac{1}{2} C_0$$

$$K=1 \Rightarrow C_3 = \frac{C_1 + C_0}{3 \cdot 2} = \frac{1}{3 \cdot 2} C_0$$

$$K=2 \Rightarrow C_4 = \frac{C_2 + C_1}{4 \cdot 3} = \frac{C_2}{4 \cdot 3} = \frac{1}{4 \cdot 3 \cdot 2} C_0$$

$$K=3 \Rightarrow C_5 = \frac{C_3 + C_2}{5 \cdot 4} = \frac{C_3}{5 \cdot 4} + \frac{C_2}{5 \cdot 4} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} C_0 + \frac{C_0}{5 \cdot 4 \cdot 2}$$

$$\Rightarrow C_5 = \frac{C_0 + 3C_0}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{4C_0}{5 \cdot 4 \cdot 3 \cdot 2}$$

and so on ...

Case 2:

Now we choose $C_0 = 0$, $C_1 \neq 0$.

$$C_2 = 0 \quad \because C_0 = 0$$

$$K=1 \Rightarrow C_3 = \frac{C_1 + C_0}{3 \cdot 2} = \frac{C_1}{3 \cdot 2}$$

$$K=2 \Rightarrow C_4 = \frac{C_2 + C_1}{4 \cdot 3} = \frac{C_1}{4 \cdot 3}$$

$$K=3 \Rightarrow C_5 = \frac{C_3 + C_2}{5 \cdot 4} = \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} C_1$$

and so on.

Putting the values of C_i's in (2), we get,

$$y = C_0 y_1(x) + C_1 y_2(x).$$

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Where

$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$

$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots$

are l.I solutions of ① and these solutions converge for all finite values of x.

Note:

It is not necessary to write a power series solution in summation form.

For nonpolynomial Coefficients.

Example.

Solve. $y'' + (\cos x)y = 0$

Solution.

Given that

$y'' + (\cos x)y = 0 \longrightarrow ①$

Clearly this DE has no singular point which gaurantees two power series solutions Centred at 0 and these solution Converge for $|x| < \infty$.

Subsituting $y = \sum_{n=0}^{\infty} C_n x^n \longrightarrow ②$

in ①, we get,

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + (\cos x)\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow (2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \cdots) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots) = 0$$

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$\Rightarrow (2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \cdots) + C_0 + C_1 x +$
$(C_2 - \frac{C_0}{2!})x^2 + (C_3 - \frac{C_1}{2!})x^3 + \cdots = 0$

$\Rightarrow (2C_2 + C_0) + (6C_3 + C_1)x + (12C_4 + C_2 - \frac{C_0}{2})x^2 +$
$(20C_5 + C_3 - \frac{C_1}{2})x^3 + \cdots = 0$

using identity properly, we get,

$2C_2 + C_0 = 0 \quad ; \quad 6C_3 + C_1 = 0$

$12C_4 + C_2 - \frac{C_0}{2} = 0 \; ; \; 20C_5 + C_3 - \frac{C_1}{2} = 0$

$\Rightarrow C_2 = -\frac{1}{2}C_0 \; ; \; C_3 = \frac{-1}{6}C_1$

$12C_4 - \frac{1}{2}C_0 - \frac{1}{2}C_0 = 0 \Rightarrow 12C_4 = C_0 \Rightarrow C_4 = \frac{1}{12}C_0$

$20C_5 - \frac{1}{6}C_1 - \frac{1}{2}C_1 = 0$

$\Rightarrow C_5 = \frac{1}{30}C_1$ and so on.

Using these values of C_i's in (2), we get

$y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^4 + C_5 x^5 + \cdots$

$\Rightarrow y = C_0 + C_1 x - \frac{1}{2}C_0 x^2 - \frac{1}{6}C_1 x^3 + \frac{1}{12}C_0 x^4 + \frac{1}{30}C_1 x^5 + \cdots$

$\Rightarrow y = C_0(1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots) + C_1(x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots)$

$\Rightarrow y = C_0 y_1(x) + C_1 y_2(x)$

where $y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots$

and $y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots$

are two L.I. solutions of (1).

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Question.1. $y'' + (\sin x)y = 0$

Solution.

Given that $y'' + (\sin x)y = 0$ ——— (1)

Clearly this DE has no singular point which guarantees 2 p.s.s centred at 0 and these solutions converge for $|x| < \infty$.

Subsituting $y = \sum_{n=0}^{\infty} c_n x^n$ ——— (2)

in (1), we get,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (\sin x)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow (2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\cdot (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots) = 0$$

$$\Rightarrow (2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots) + c_0 x + c_1 x^2 + \left(c_2 - \frac{c_0}{3!}\right)x^3 + \cdots$$

$$\Rightarrow 2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2 + \left(20c_5 + c_2 - \frac{c_0}{6}\right)x^3 + \cdots$$

Using identity properly, we get,

$2c_2 = 0$; $6c_3 + c_0 = 0$; $12c_4 + c_1 = 0$

$20c_5 + c_2 - \frac{c_0}{6} = 0$

$\Rightarrow c_2 = 0$; $c_3 = \frac{-1}{6}c_0$; $c_4 = -\frac{1}{12}c_1$

$c_5 = +\frac{c_0}{120}$ and so on

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Putting these values of C_i's in (2), we get

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \cdots$$

$$\Rightarrow y = C_0 + C_1 x + \left(-\frac{1}{6} C_0\right) x^3 - \frac{1}{12} C_1 x^4 + \frac{1}{120} C_0 x^5 + \cdots$$

$$\Rightarrow y = C_0 \left(1 - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \cdots\right) + C_1 \left(x - \frac{1}{12} x^5 + \cdots\right)$$

$$y = C_0 y_1(x) + C_1 y_2(x)$$

where $y_1(x) = 1 - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \cdots$

and $y_2(x) = x - \frac{1}{12} x^5 + \cdots$

are two l.I solutions of (1). These solutions converge for $|x| < \infty$.

Question 2: $y'' + e^x y' - y = 0$

Solution.

Given that $y'' + e^x y' - y = 0$ ——— (1)

Clearly this DE has no singular point which guarantees two power series solutions c.at 0 and these solutions converge for $|x| < \infty$.

Subsituting $y = \sum_{n=0}^{\infty} C_n x^n$ ——— (2) in (1), we get,

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + e^x \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow (2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \cdots) + \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$
$$(C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \cdots) - (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots) = 0$$

$$\Rightarrow (2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \cdots) + \left(C_1 + (2C_2 + C_1) x + \left(3C_3 + 2C_2 + \frac{C_1}{2}\right) x^2 + \left(4C_4 + 3C_3 + 2C_2 + \frac{C_1}{6}\right) x^3 + \cdots\right) - (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots) = 0$$

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$\Rightarrow (2C_2+C_1-C_0)+(6C_3+2C_2)x+(12C_4+4C_2+\frac{C_1}{2})x^2$
$+(20C_5+4C_4+3C_3+2C_2+\frac{C_1}{6}-C_3)x^3+\cdots=0$

Using identity properly, we get,

$2C_2+C_1-C_0=0 \;;\; 6C_3+2C_2=0$

$12C_4+4C_2+\frac{C_1}{2}=0 \;;\; 20C_5+4C_4+2C_3+2C_2+\frac{C_1}{6}=0$

$\Rightarrow C_2=\frac{C_0-C_1}{2} \;;\; C_3=\frac{-1}{3}C_2=\frac{C_1-C_0}{6}$

$12C_4+2(C_0-C_1)+\frac{C_1}{2}=0 \Rightarrow 12C_4+2C_0=\frac{3}{2}C_1$

$\Rightarrow C_4=\frac{3C_1-4C_0}{24}$

Putting these values of C_i's in (2), we get

$y=C_0+C_1x+\frac{C_0-C_1}{2}x^2+\frac{C_1-C_0}{3\cdot 2}x^3+\frac{3C_1-4C_0}{24}x^4$

$+\cdots$

$=C_0+C_1x+\frac{C_0}{2}x^2-\frac{C_1}{2}x^2+\frac{C_1}{6}x^3-\frac{C_0}{6}x^3$

$+\frac{C_1}{8}x^4-\frac{C_0}{6}x^4+\cdots$

$=C_0\left(1+\frac{x^2}{2}-\frac{x^3}{6}-\frac{x^4}{6}+\cdots\right)+C_1\left(x-\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots\right)$

$\Rightarrow y=C_0y_1(x)+C_1y_2(x)$

where

$y_1(x)=1+\frac{x^2}{2}-\frac{x^3}{6_3}-\frac{x^4}{6_4}+\cdots$

and $y_2(x)=x-\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{8}+\cdots$ are two L.I. solutions of (1).

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Regular Singular Point. 2013, 2015,

A singular point x_0 is said to be a regular singular point of the diff. equation $y'' + P(x)y' + Q(x)y = 0$ if the functions $p(x) = (x-x_0)P(x)$ and $q(x) = (x-x_0)^2 Q(x)$ are both analytic at x_0.

e.g. $xy'' + y = 0$ ——— (i)

$\Rightarrow y'' + \frac{1}{x}y = 0$

Here $P(x) = 0$ and $Q(x) = 1/x$

$p(x) = 0$ and $q(x) = x$

$x = 0$ is a regular singular point of (i).

Irregular Singular Point. 2013, 2015,

A singular point x_0 is said to be an irregular point of the DE $y'' + P(x)y' + Q(x)y = 0$ if one or both of the functions $p(x) = (x-x_0)P(x)$ and $q(x) = (x-x_0)^2 Q(x)$ fails to be analytic at x_0.

e.g. $x^3y'' + y = 0$ ——— (ii)

$\Rightarrow y'' + \frac{1}{x^3}y = 0$

Here $P(x) = 0$ and $Q(x) = 1/x^3$

$p(x) = 0$ and $q(x) = (x-0)\cdot 1/x^3$

$= 1/x$

$x = 0$ is an irregular singular point of (ii).

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Frobenius Theorem.

If $x = x_0$ is a regular singular point of the DE $y'' + P(x)y' + Q(x)y = 0$, then there exists at least one solution of the form $y = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$

where the number r is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Note:

If r is found to be a number that is not a non negative integer ($r \in \mathbb{R} \setminus \{0, 1, 2, \ldots\}$) then the corresponding solution $y = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$ is not a power series.

Example.

Solve $3xy'' + y' - y = 0$.

Solution.

Given that $3xy'' + y' - y = 0$ ——— (1)

Therefore, $y'' + \frac{1}{3x} y' - \frac{1}{3x} y = 0$

Here $P(x) = 1/3x$ and $Q(x) = -1/3x$

$p(x) = x \cdot P(x) = 1/3$ and $q(x) = x^2 Q(x)$

$= -x/3$

Clearly $x = 0$ is a regular singular point of (1).

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Now we find a solution of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+s} \longrightarrow (2)$$

$$y' = \sum_{n=0}^{\infty} (n+s) C_n x^{n+s-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s-2}$$

Putting the values of y, y' and y'' in (1), we have,

$$3x \sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s-2} + \sum_{n=0}^{\infty} (n+s) C_n x^{n+s-1} - \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow 3 \sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) C_n x^{n+s-1} - \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [3(n+s)(n+s-1) + (n+s)] C_n x^{n+s-1} - \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} [(n+s)(3n+3s-3+1)] C_n x^{n+s-1} - \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow x^s \left[\sum_{n=0}^{\infty} (n+s)(3n+3s-2) C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n \right] = 0$$

$$\Rightarrow x^s \left[s(3s-2) C_0 x^{-1} + \sum_{n=1}^{\infty} (n+s)(3n+3s-2) C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n \right] = 0$$

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$$\Rightarrow x^{s}\left[s(3s-2)c_0x^{-1}+\sum_{k=0}^{\infty}(k+1+s)(3k+3+3s-2)c_{k+1}x^{k}-\sum_{k=0}^{\infty}c_kx^{k}\right]=0$$

$$\Rightarrow x^{s}\left[s(3s-2)c_0x^{-1}+\sum_{k=0}^{\infty}(k+s+1)(3k+3s+1)c_{k+1}x^{k}-\sum_{k=0}^{\infty}c_kx^{k}\right]=0$$

$$\Rightarrow x^{s}\left[s(3s-2)c_0x^{-1}+\sum_{k=0}^{\infty}\{(k+s+1)(3k+3s+1)c_{k+1}-c_k\}x^{k}\right]=0$$

Using identity properly, we get,

$s(3s-2)c_0=0$; $(k+s+1)(3k+3s+1)c_{k+1}-c_k=0$

Since for $c_0=0$ we proceed to the trivial solution which is not our target so $c_0\neq 0$.

$\Rightarrow s(3s-2)=0$

$\Rightarrow s=0$ or $(3s-2)=0$

say: $s_2=0$; $s_1=2/3$

and

$$c_{k+1}=\frac{c_k}{(k+s+1)(3k+3s+1)}\quad ;\ k=0,1,2,\cdots$$

For $s_2=0$

$$c_{k+1}=\frac{c_k}{(k+1)(3k+1)}\quad ;\ k=0,1,2,\cdots \longrightarrow (3)$$

For $s_1=2/3$

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$$C_{k+1} = \frac{C_k}{\left(\frac{3k+5}{3}\right)(3k+3)} = \frac{C_k}{(3k+5)(k+1)} \longrightarrow (4)$$

From (3): $k=0 \Rightarrow C_1 = \frac{C_0}{1\cdot 1} = C_0$

$$k=1 \Rightarrow C_2 = \frac{C_1}{2\cdot 4} = \frac{C_0}{2!\ 1\cdot 4}$$

$$k=2 \Rightarrow C_3 = \frac{C_2}{3\cdot 7} = \frac{C_0}{3!\ 1\cdot 4\cdot 7}$$

$$k=3 \Rightarrow C_4 = \frac{C_3}{4\cdot 10} = \frac{C_0}{4!\ 1\cdot 4\cdot 7\cdot 10}$$

and so on.

$$C_n = \frac{C_0}{n!\ 1\cdot 4\cdot 7 \cdots (3n-2)}$$

From (4):

$k=0 \Rightarrow C_1 = \frac{C_0}{1\cdot 5}$; $k=1 \Rightarrow C_2 = \frac{C_1}{2\cdot 8} = \frac{C_0}{2!\ 5\cdot 8}$

$$k=2 \Rightarrow C_3 = \frac{C_2}{3\cdot 11} = \frac{C_0}{3!\ 5\cdot 8\cdot 11}$$

$$k=3 \Rightarrow C_4 = \frac{C_3}{4\cdot 14} = \frac{C_0}{4!\ 5\cdot 8\cdot 11\cdot 14}$$

and so on.

$$C_n = \frac{C_0}{n!\ 5\cdot 8\cdot 11\cdot 14 \cdots (3n+2)}$$

Thus (2) implies that

$$y_1(x) = x^0\left[C_0 + \frac{C_0}{1\cdot 1}x + \frac{C_0}{2!\ 1\cdot 4}x^2 + \frac{C_0}{3!\ 1\cdot 4\cdot 7}x^3 + \cdots\right]$$

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$$= C_0\left(1 + \frac{1}{1\cdot 1}x + \frac{1}{2!\,1\cdot 4}x^2 + \frac{1}{3!\,1\cdot 4\cdot 7}x^3 + \cdots\right)$$

$$\Rightarrow y_1(x) = C_0\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!\,1\cdot 4\cdot 7\cdots(3n-2)}x^n\right) \longrightarrow (5)$$

Thus (2) implies that

$$y_2(x) = C_1 x^{2/3}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n!\,5\cdot 8\cdot 11\cdots(3n+2)}x^n\right) \rightarrow (6)$$

From (5) and (6) the general solution is $y = C_0 y_1(x) + C_1 y_2(x)$ and these solutions $y_1(x)$ and $y_2(x)$ converge for $0 < x < \infty$.

Three cases.

Suppose that $x=0$ is a regular singular point of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ and that ~~that~~ indicial roots s_1 and s_2 of the singularity are real, with s_1 denoting the largest root. When using the method of Frobenius, we distinguish three cases corresponding to the nature of indicial roots s_1 and s_2.

Case 1:

If s_1 and s_2 are distinct and do not differ by an integer then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} C_n x^{n+s_1}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+s_2}$$

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Case 2:

If $s_1 - s_2 = N$ where N is a +ive integer then there exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+s_1} \quad ; c_0 \neq 0$$

$$y_2(x) = C y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+s_2} \quad ; b_0 \neq 0.$$

where C is a constant that could be zero. When the difference of indicial roots $s_1 - s_2$ is a +ive integer, the 2nd solution may contain a logarithm.

Case 3:

If $s_1 = s_2$ then there always exist two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+s_1} \quad ; c_0 \neq 0$$

and $$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+s_2}$$

When the Frobenius method fails to give a 2nd series solution, we obtain the 2nd solution with the logarithmic term by the following relation.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx$$

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Remark:

① If $s_1 - s_2 = 1$, then Frobenius method fails to find 2nd solution.

② If $x=0$ is an irregular singular point, we may not be able to find any solution of the DE of form $y = \sum_{n=0}^{\infty} c_n x^{n+s}$.

Example: 2013, Special

Solve $2xy'' + (1+x)y' + y = 0$

Solution.

Given that $2xy'' + (1+x)y' + y = 0 \longrightarrow (1)$

Therefore, $y'' + \left(\frac{1+x}{2x}\right)y' + \frac{1}{2x}y = 0$

Clearly $x=0$ is a regular singular point of (1). Now we find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+s} \longrightarrow (2)$$

Putting the values of y, y' and y'' in (1), we get

$$2\sum_{n=0}^{\infty}(n+s)(n+s-1)c_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)c_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)c_n x^{n+s} + \sum_{n=0}^{\infty} c_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+s)(2n+2s-2+1)c_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s+1)c_n x^{n+s} = 0$$

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$$\Rightarrow x^s\left[\sum_{n=0}^{\infty}(n+s)(2n+2s-1)C_n x^{n+s-1}+\sum_{n=0}^{\infty}(n+s+1)C_n x^n\right]=0$$

$$\Rightarrow x^s\left[s(2s-1)C_0 x^{-1}+\sum_{n=1}^{\infty}(n+s)(2n+2s-1)C_n x^{n-1}+\sum_{n=0}^{\infty}(n+s+1)C_n x^n\right]=0$$

$$\Rightarrow x^s\left[s(2s-1)C_0 x^{-1}+\sum_{K=0}^{\infty}(K+1+s)(2K+2s+1)C_{K+1}x^K+\sum_{K=0}^{\infty}(K+s+1)C_K x^K\right]=0$$

$$\Rightarrow x^s\left[s(2s-1)C_0 x^{-1}+\sum_{K=0}^{\infty}(K+s+1)\{(2K+2s+1)C_{K+1}+C_K\}x^K\right]=0$$

using identity properly, we get

$$s(2s-1)C_0=0 \quad ; \quad (K+s+1)\{(2K+2s+1)C_{K+1}+C_K\}=0$$

$$\Rightarrow s=0 \text{ or } s=\frac{1}{2} \quad ; \quad C_{K+1}=\frac{-C_K}{2K+2s+1}$$

(Say) $s=s_2=0$

$$C_{K+1}=\frac{-C_K}{2K+1} \longrightarrow (3)$$

(Say) $s=\frac{1}{2}$

$$C_{K+1}=\frac{-C_K}{2K+1+1}=\frac{-C_K}{2K+2}$$

$$\Rightarrow C_{K+1}=\frac{-C_K}{2(K+1)} \longrightarrow (4)$$

From (3),

$C_1=-C_0$

$C_2=\frac{-C_1}{3}=\frac{C_0}{1\cdot 3}$

$C_3=\frac{-C_2}{5}=\frac{C_0}{1\cdot 3\cdot 5}$

From (4)

$C_1=\frac{-C_0}{2\cdot 1}$

$C_2=\frac{-C_1}{2\cdot 2}=\frac{C_0}{2^2\cdot 2!}$

$C_3=\frac{-C_2}{2\cdot 3}=\frac{-C_0}{2^3\cdot 3!}$

and so on.

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For indicial root $s_2 = 0$, we obtain a solution

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^n \quad . |x| < \infty$$

For indicial root $s_1 = 1/2$, we obtain a solution

$$y_2(x) = x^{1/2}\left\{1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n\right\}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+\frac{1}{2}}.$$

Thus on $(0, \infty)$ the general solution is

$$y = C_1 y_1(x) + C_2 y_2(x).$$

Question 1: 2018, 2014, Special

Solve. $2xy'' - y' + 2y = 0$

Solution.

Given that $2xy'' - y' + 2y = 0$ ——— (1)

Therefore, $y'' - \frac{1}{2x} y' + \frac{1}{x} y = 0$

Clearly $x = 0$ is a regular singular point of (1). Now we find the solution of (1) of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+s} \longrightarrow (2)$$

$$(1) \Rightarrow 2\sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s) C_n x^{n+s-1}$$

$$+ 2\sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(2n+2s-2-1) C_n x^{n+s-1} + 2\sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

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$$\Rightarrow x^{s}\left[\sum_{n=0}^{\infty}(n+s)(2n+2s-3)C_n x^{n+s-2}+2\sum_{n=0}^{\infty}C_n x^n\right]=0$$

$$\Rightarrow x^{s}\left[s(2s-3)C_0 x^{-1}+\sum_{n=1}^{\infty}(n+s)(2n+2s-3)C_n x^{n-1}+2\sum_{n=0}^{\infty}C_n x^n\right]=0$$

$$\Rightarrow x^{s}\left[s(2s-3)C_0 x^{-1}+\sum_{k=0}^{\infty}(k+1+s)(2k+2s-1)C_{k+1}x^{k}+2\sum_{k=0}^{\infty}C_k x^k\right]=0$$

$$\Rightarrow x^{s}\left[s(2s-3)C_0 x^{-1}+\sum_{k=0}^{\infty}\left\{(k+1+s)(2k+2s-1)C_{k+1}+2C_k\right\}x^k\right]=0$$

using identity properly, we have.

$s(2s-3)C_0=0$; $(k+1+s)(2k+2s-1)C_{k+1}=-2C_k$

$\Rightarrow s=0; s=\frac{3}{2}$; $C_{k+1}=\dfrac{-2C_k}{(k+1+s)(2k+2s-1)}$

(Say) $s=s_1=3/2 \Rightarrow C_{k+1}=\dfrac{-2C_k}{(k+1+\frac{3}{2})(2k+3-1)}$

$\Rightarrow C_{k+1}=\dfrac{-2C_k}{\frac{(2k+2+3)}{2}2(k+1)}=\dfrac{-2C_k}{(2k+5)(k+1)}$ ——— (3)

(Say) $s_2=0$; $\Rightarrow C_{k+1}=\dfrac{-2C_k}{(k+1)(2k-1)}$ ——— (4)

From (3), we can write,

$K=0 \Rightarrow C_1=\dfrac{-2C_0}{(1)(5)}=\dfrac{-2C_0}{5\cdot 1}$

$K=1 \Rightarrow C_2=\dfrac{-2C_1}{7\cdot 2}=\dfrac{4C_0}{7\cdot 5\cdot 2\cdot 1}$

$K=2 \Rightarrow C_3=\dfrac{-2C_2}{9\cdot 3}=\dfrac{-8C_0}{9\cdot 7\cdot 5\cdot 3\cdot 2\cdot 1}$

and so on

From (4).

$C_1=\dfrac{-2C_0}{(1)(-1)}=+2C_0$

$C_2=\dfrac{-2C_1}{2\cdot 1}=-2C_1$

$=-2C_0$

$C_3=\dfrac{-2C_2}{3\cdot 3}=\dfrac{-2C_2}{9}$

$=+\dfrac{4C_0}{9}$

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For $s_1 = 3/2$ ② implies that

$$y_1(x) = x^{3/2}\left[c_0 - \frac{2c_1}{5\cdot1}x + \frac{4c_0}{7\cdot5\cdot2\cdot1}x^2 - \frac{8c_0}{9\cdot7\cdot5\cdot3\cdot2\cdot1}x^3 + \cdots\right]$$

$$\Rightarrow y_1(x) = c_0 x^{3/2}\left(1 - \frac{2}{5\cdot1}x + \frac{4}{7\cdot5\cdot2\cdot1}x^2 - \frac{8}{9\cdot7\cdot5\cdot3\cdot2\cdot1}x^3 + \cdots\right)$$

For $s_2 = 0$ ② implies that

$$y_2(x) = x^0\left[c_0 + 2c_0x - 2c_0x^2 + \frac{4c_0}{9}x^3 - \cdots\right]$$

$$y_2(x) = c_0x^0\left[1 + 2x - 2x^2 + \frac{4}{9}x^3 - \cdots\right]$$

Thus on the interval $(0,\infty)$ the general solution is

$$y = c_0y_1(x) + c_0y_2(x).$$

Question 2:

Solve: $x^2y'' - (x - \frac{2}{9})y = 0$

Solution.

Given that $x^2y'' - (x - \frac{2}{9})y = 0$ — ①

Therefore, $y'' - \frac{1}{x^2}(x - \frac{2}{9})y = 0$

Clearly $x = 0$ is a regular singular point of ①. Now we find the solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+s} \quad \text{——} \quad ②$$

$$① \Rightarrow \sum_{n=0}^{\infty}(n+s)(n+s-1)c_nx^{n+s} - \sum_{n=0}^{\infty}c_nx^{n+s+1} + \frac{2}{9}\sum_{n=0}^{\infty}c_nx^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}\left((n+s)(n+s-1) + \frac{2}{9}\right)c_nx^{n+s} - \sum_{n=0}^{\infty}c_nx^{n+s+1} = 0$$

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$$\Rightarrow x^{s}\left[\sum_{n=0}^{\infty}\left((n+s)(n+s-1)+\frac{2}{9}\right)C_n x^n-\sum_{n=0}^{\infty}C_n x^{n+1}\right]=0$$

$$\Rightarrow x^{s}\left[\left(s(s-1)+\frac{2}{9}\right)C_0+\sum_{n=1}^{\infty}\left((n+s)(n+s-1)+\frac{2}{9}\right)C_n x^n - \sum_{n=0}^{\infty}C_n x^{n+1}\right]=0$$

$$\Rightarrow x^{s}\left[\left(s(s-1)+\frac{2}{9}\right)C_0+\sum_{k=1}^{\infty}\left((k+s)(k+s-1)+\frac{2}{9}\right)C_k x^k - \sum_{k=1}^{\infty}C_{k-1}x^k\right]=0$$

$$\Rightarrow x^{s}\left[\left(s^2-s+\frac{2}{9}\right)C_0+\sum_{k=1}^{\infty}\left\{\left((k+s)(k+s-1)+\frac{2}{9}\right)C_k - C_{k-1}\right\}x^k\right]=0$$

Using identity properly, we get,

$\left(s^2-s+\frac{2}{9}\right)C_0=0$; $\left((k+s)(k+s-1)+\frac{2}{9}\right)C_k = C_{k-1}$

Since for $C_0=0$, we proceed to the trivial solution which is not our target

so $C_0 \neq 0$.

$s^2-s+\frac{2}{9}=0 \Rightarrow 9s^2-9s+2=0$

$\Rightarrow 9s^2-6s-3s+2=0 \Rightarrow s=2/3,\ 1/3$

(Say) $s_1 = 2/3$; $C_k = \dfrac{C_{k-1}}{(k+s)(k+s-1)+\frac{2}{9}}$

$$\Rightarrow C_k = \frac{C_{k-1}}{\left(k+\frac{2}{3}\right)\left(k+\frac{2}{3}-1\right)+\frac{2}{9}}$$

$$\Rightarrow C_k = \frac{9C_{k-1}}{(3k+2)(3k-1)+2} \qquad \text{---------- (3)}$$

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(Say) $r_2 = 1/3$;

$$C_K = \frac{9C_{K-1}}{(3K+1)(3K-2)+2} \longrightarrow (4)$$

From (3); From (4)

$K=1 \Rightarrow C_1 = \frac{3}{4}C_0$; $C_1 = \frac{9C_0}{4 \cdot 1 + 2} = \frac{9}{6}C_0 = \frac{3}{2}C_0$

$K \Rightarrow 2 \Rightarrow C_2 = \frac{9}{56}C_0$; $C_2 = \frac{9}{30}C_0$

$k=3 \Rightarrow C_3 = \frac{9}{560}C_0$; $C_3 = \frac{9}{160}C_0$

$a_n \longrightarrow$ so on.

For $r_1 = 2/3$ (2) implies that

$$y_1 = x^{2/3}\left(C_0 + \frac{3}{4}C_0x + \frac{9}{56}C_0x^2 + \frac{9}{560}C_0x^3 + \cdots\right)$$

$$\Rightarrow y_1(x) = x^{2/3}C_0\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)$$

For $r_2 = 1/3$ (2) implies that

$$y_2(x) = x^{1/3}\left(C_0 + \frac{3}{2}C_0x + \frac{9}{20}C_0x^2 + \frac{9}{160}x^3 + \cdots\right)$$

$$\Rightarrow y_2(x) = C_0x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right)$$

Thus the general solution on $(0, \infty)$ is

$$y = C_1y_1(x) + C_0y_2(x)$$

where

$$y_1(x) = x^{2/3}\left(1 + \frac{3}{4}x + \frac{9}{56}x^2 + \frac{9}{560}x^3 + \cdots\right)$$

$$y_2(x) = x^{1/3}\left(1 + \frac{3}{2}x + \frac{9}{20}x^2 + \frac{9}{160}x^3 + \cdots\right).$$

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Solve the following by using Frobenius method.

(3) $4xy'' + \frac{1}{2}y' + y = 0$; (4) $3xy'' + (2-x)y' - y = 0$

(5) $2xy'' - (3+2x)y' + y = 0$; (6) $9x^2y'' + 9x^2y' + 2y = 0$

(7) $2x^2y'' + 3xy' + (2x-1)y = 0$.

Example:

Solve $xy'' + y = 0$

Solution.

Given that $xy'' + y = 0$ ——— (1)

Therefore, $y'' + \frac{1}{x}y = 0$

Clearly $x = 0$ is a regular singular point of (1). Now we find the solution of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+s} \quad \text{——— (2)}$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s-1} + \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow x^s \left[\sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n \right] = 0$$

$$\Rightarrow x^s \left[s(s-1) C_0 x^{-1} + \sum_{n=1}^{\infty} (n+s)(n+s-1) C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^n \right] = 0$$

$$\Rightarrow x^s \left[s(s-1) C_0 x^{-1} + \sum_{k=0}^{\infty} (k+1+s)(k+s) C_{k+1} x^k + \sum_{k=0}^{\infty} C_k x^k \right] = 0$$

$$\Rightarrow x^s \left[s(s-1) C_0 x^{-1} + \sum_{k=0}^{\infty} \{ (k+s)(k+s+1) C_{k+1} + C_k \} x^k \right] = 0$$

using identity properly, we get,

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$r(r-1)c_0 = 0$; $(k+r)(k+r+1)c_{k+1} + c_k = 0$

Since for $c_0 = 0$, we proceed to the trivial solution which is not our target so $c_0 \neq 0$.

$\Rightarrow r(r-1) = 0 \Rightarrow r = 0$ or $r = 1$

(say) $r = r_1 = 1$

$$c_{k+1} = \frac{-c_k}{(k+r)(k+r+1)}$$ becomes

$$c_{k+1} = \frac{-c_k}{(k+1)(k+2)} \longrightarrow (3)$$

(say) $r = r_2 = 0$.

so $$c_{k+1} = \frac{-c_k}{k(k+1)} \quad \text{---} (4)$$

From (3).

$k = 0 \Rightarrow c_1 = \frac{-c_0}{1 \cdot 2} = \frac{-c_0}{2}$

$k = 1 \Rightarrow c_2 = \frac{-c_1}{2 \cdot 3} = \frac{-c_1}{6} = \frac{c_0}{12}$

$k = 2 \Rightarrow c_3 = \frac{-c_2}{3 \cdot 4} = \frac{-c_2}{12} = \frac{-c_0}{144}$

and so on.

$$(2) \Rightarrow y_1(x) = x^r\left[c_0 - \frac{c_0}{2}x + \frac{c_0}{12}x^2 - \frac{c_0}{144}x^3 + \cdots\right]$$

$$= c_0\left(x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} + \cdots\right)$$

Since $r_1 - r_2 = 1$, Frobenius method fails to give 2nd series solution.

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In this case Frobenius method produces only one series solution, we find the 2nd series solution by the formula.

$$y_2(x) = y_1(x)\int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}\,dx$$

$$\Rightarrow y_2(x) = y_1(x)\int \frac{dx}{\left(x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\cdots\right)^2}$$

$\{P(x)=0, \int 0\,dx = 0, e^0 = 1\}$

$$\left(x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\cdots\right)\left(x-\frac{x^2}{2}+\frac{x^3}{12}-\frac{x^4}{144}+\cdots\right)$$

$$= x^2 - \frac{1}{2}x^3 + \frac{1}{12}x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^4 - \frac{1}{24}x^5 + \frac{1}{12}x^4 - \frac{1}{24}x^5 + \cdots$$

$$= x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{12}x^5 + \cdots$$

So

$$y_2(x) = y_1(x)\int \frac{dx}{x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{12}x^5 + \cdots}$$

$$= y_1(x)\int \left(x^2 - x^3 + \frac{5}{12}x^4 - \frac{7}{12}x^5 + \cdots\right)^{-1} dx$$

$$= y_1(x)\int x^{-2}\left(1 - x + \frac{5}{12}x^2 - \frac{7}{12}x^3 + \cdots\right)^{-1} dx$$

$$= y_1(x)\int \frac{1}{x^2}\left(1 + \left(-x + \frac{5}{12}x^2 - \frac{7}{12}x^3 + \cdots\right)\right)^{-1} dx$$

Binomial

$$= y_1(x)\int \frac{1}{x^2}\left(1 + x - \frac{5}{12}x^2 + \frac{7}{12}x^3 + x^2 + \frac{25}{144}x^4 + \frac{49}{144}x^6 + x^3 + \cdots\right) dx$$

$$= y_1(x)\int \frac{1}{x^2}\left(1 + x + \frac{7}{12}x^2 + \frac{19}{12}x^3 + \cdots\right) dx$$

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$$= y_1(x)\int\left(\frac{1}{x^2}+\frac{1}{x}+\frac{7}{12}+\frac{19}{72}x+\cdots\right)dx$$

$$= y_1(x)\left(-\frac{1}{x}+\ln x+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\right)$$

$$y_2(x) = y_1(x)\ln x + y_1(x)\left(-\frac{1}{x}+\frac{7}{12}x+\frac{19}{144}x^2+\cdots\right)$$

Hence, on the interval $(0,\infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$.

Question: 2015, Special

Solve, $xy'' + (1-x)y' - y = 0$

Solution.

Given that $xy'' + (1-x)y' - y = 0$

$$xy'' + y' - xy' - y = 0 \quad \text{——} (1)$$

So $$y'' + \frac{1}{x}y' - y' - \frac{1}{x}y = 0$$

Clearly $x=0$ is a regular singular pt of (1). Now we find the solution of the form

$$y(x) = \sum_{n=0}^{\infty} C_n x^{n+s} \quad \text{——} (2)$$

$$(1) \Rightarrow \sum_{n=0}^{\infty}(n+s)(n+s-1)C_n x^{n+s-1} + \sum_{n=0}^{\infty}(n+s)C_n x^{n+s-1}$$

$$-\sum_{n=0}^{\infty}(n+s)C_n x^{n+s} - \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+s)(n+s-1+1)C_n x^{n+s-1} - \sum_{n=0}^{\infty}(n+s+1)C_n x^{n+s} = 0$$

$$\Rightarrow x^s\left[\sum_{n=0}^{\infty}(n+s)^2 C_n x^{n-1} - \sum_{n=0}^{\infty}(n+s+1)C_n x^n\right] = 0$$

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$$\Rightarrow x^{r}\left[r^2 c_0 x^{-1} + \sum_{n=1}^{\infty}(n+r)^2 c_n x^{n-1} - \sum_{n=0}^{\infty}(n+r+1)c_n x^{n}\right]=0$$

$$\Rightarrow x^{r}\left[r^2 c_0 x^{-1} + \sum_{k=0}^{\infty}(k+1+r)^2 c_{k+1} x^{k} - \sum_{k=0}^{\infty}(k+r+1)c_k x^{k}\right]=0$$

$$\Rightarrow x^{r}\left[r^2 c_0 x^{-1} + \sum_{k=0}^{\infty}(k+r+1)\{(k+r+1)c_{k+1} - c_k\}x^{k}\right]=0$$

∴ by using identity properly, we get,

$r^2 c_0 = 0$; $(k+r+1)c_{k+1} = c_k$

Since, for $c_0 = 0$, we proceed to a trivial solution which is not our target, so $c_0 \neq 0$.

$\Rightarrow r = 0$; $c_{k+1} = \dfrac{c_k}{k+r+1}$

$$\Rightarrow c_{k+1} = \frac{c_k}{k+1}, \quad k = 0, 1, 2, \text{---}.$$

Here $r_1 = 0$, $r_2 = 0$ give same set of co-efficients

$k=0 \Rightarrow c_1 = \dfrac{c_0}{1} = c_0$

$k=1 \Rightarrow c_2 = \dfrac{c_1}{2} = \dfrac{c_0}{2}$

$k=2 \Rightarrow c_3 = \dfrac{c_2}{3} = \dfrac{c_0}{6}$

$k=3 \Rightarrow c_4 = \dfrac{c_3}{4} = \dfrac{c_0}{24}$ and so on.

From (2):

$$y_1(x) = x^0\left[c_0 + c_0 x + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \cdots\right]$$

$$= c_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)$$

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$y_1(x) = c_0 e^x$

Frobenius method give one series solution because $r_1 = r_2$.

Now we find the 2nd series solution by following formula,

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2} dx \quad \text{——— (3)}$$

From (1):

$$P(x) = \left(\frac{1-x}{x}\right) = \frac{1}{x} - 1,$$

$$\int P(x)dx = \int \left(\frac{1}{x} - 1\right)dx = \ln x - x$$

Substituting in (3), we get,

$$y_2(x) = e^x \int \frac{e^{x - \ln x}}{e^{2x}} dx$$

$$= e^x \int \frac{e^{x-\ln x}}{e^x \cdot e^x} dx = e^x \int \frac{e^x \cdot e^{-\ln x}}{e^x \cdot e^x} dx$$

$$= e^x \int \frac{e^{-\ln x}}{e^x} dx = e^x \int \frac{e^{\ln x^{-1}}}{e^x} dx$$

$$= e^x \int \frac{e^{\ln(1/x)}}{e^x} dx = e^x \int \frac{1/x}{e^x} dx$$

$$= e^x \int \frac{dx}{xe^x} = e^x \int \frac{dx}{x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)}$$

$$= e^x \int \frac{dx}{x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots}$$

$$= e^x \int \left(\frac{1}{x} - 1 + \frac{x}{2!} - \frac{x^2}{3!} + \cdots\right)dx$$ by binomial.

or e^{-x} =

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$$= e^x\left(\ln x - x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \cdots\right)$$

$$= e^x \ln x - e^x\left(x + \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \cdots\right)$$

$$y_2(x) = e^x \ln x - e^x \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\cdot n!} x^n$$

Hence the general solution on $(0,\infty)$ is $y(x) = C_1 y(x) + C_2 y_2(x)$.

Solve: ② $xy'' - xy' + y = 0$

③ $y'' + \frac{3}{x}y' - 2y = 0$ ④ $xy'' + y' + y = 0$.

Question:

Solve $2xy'' + 5y' + xy = 0$

Solution:

Given that $2xy'' + 5y' + xy = 0$ ——— ①

Therefore, $y'' + \frac{5}{2x}y' + \frac{1}{2}y = 0$

Clearly $x=0$ is a regular singular point of ①. Now, we find the solution of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+r} \longrightarrow ②$$

$$① \Rightarrow 2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-1} + 5\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r+1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty}(n+r)(2n+2r-2+5)C_n x^{n+r-1} + \sum_{n=0}^{\infty} C_n x^{n+r+1} = 0$$

$$\Rightarrow x^r\left[\sum_{n=0}^{\infty}(n+r)(2n+2r+3)C_n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1}\right] = 0.$$

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$$\Rightarrow x^{s}\left[s(2s+3)c_0x^{-1}+(1+s)(2s+5)c_1+\sum_{n=2}^{\infty}(n+s)(2n+2s+3)c_nx^{n-1}+\sum_{n=0}^{\infty}c_nx^{n+1}\right]=0$$

$$\Rightarrow x^{s}\left[s(2s+3)c_0x^{-1}+(1+s)(2s+5)c_1+\sum_{k=1}^{\infty}(k+1+s)(2k+2s+5)c_{k+1}x^{k}+\sum_{k=1}^{\infty}c_{k-1}x^{k}\right]=0$$

$$\Rightarrow x^{s}\left[s(2s+3)c_0x^{-1}+(1+s)(2s+5)c_1+\sum_{k=1}^{\infty}\{(k+s+1)(2k+2s+5)c_{k+1}+c_{k-1}\}x^{k}\right]=0$$

using identity property, we get,

$s(2s+3)c_0=0$; $(1+s)(2s+5)c_1=0$

$(k+s+1)(2k+2s+5)c_{k+1}=-c_{k-1}$

Since for $c_0=0$, we proceed to a trivial solution, which is not our target so $c_0\neq 0$.

$s(2s+3)=0$

$\Rightarrow s=0$ or $s=-3/2$

$\Rightarrow c_1=0$

(say) $s=s_1=0$

then $c_{k+1}=\dfrac{-c_{k-1}}{(k+1)(2k+5)}$, $k=1,2,3,\ldots$ ⟶ (3)

(say) $s=s_2=-3/2$

$$c_{k+1}=\frac{-c_{k-1}}{(k-\frac{1}{2})(k+2)}=\frac{-2c_{k-1}}{(2k-1)(k+2)}$$ ⟶ (4)

We find the general solution from (2) and (4) in the form of (2). as

$y=c_1y_1(x)+c_2y_2(x)$.

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Question:

Solve $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

Solution.

Given that $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ — (1)

so $x^2y'' + xy' + x^2y - \frac{1}{4}y = 0$

Clearly $x=0$ is a regular singular point of (1). Now we find the solution of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+k} \quad \text{——— (2)}$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+k)(n+k-1) C_n x^{n+k} + \sum_{n=0}^{\infty} (n+k) C_n x^{n+k}$$

$$+ \sum_{n=0}^{\infty} C_n x^{n+k+2} - \frac{1}{4} \sum_{n=0}^{\infty} C_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+k)\{(n+k-1)+1\} C_n x^{n+k} + \sum_{n=0}^{\infty} C_n x^{n+k+2} - \frac{1}{4} \sum_{n=0}^{\infty} C_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+k)^2 - \frac{1}{4} \right] C_n x^{n} + \sum_{n=0}^{\infty} C_n x^{n+2} = 0$$

$$\Rightarrow x^k \left[\sum_{n=0}^{\infty} \left[(n+k)^2 - \frac{1}{4} \right] C_n x^{n} + \sum_{n=0}^{\infty} C_n x^{n+2} \right] = 0$$

$$\Rightarrow x^k \left[\left(k^2 - \frac{1}{4}\right) C_0 x^0 + \left((1+k)^2 - \frac{1}{4}\right) C_1 x + \sum_{n=2}^{\infty} \left[(n+k)^2 - \frac{1}{4} \right] C_n x^{n} \right.$$

$$\left. + \sum_{n=0}^{\infty} C_n x^{n+2} \right] = 0$$

$$\Rightarrow x^k \left[\left(k^2 - \frac{1}{4}\right) C_0 + \left(k^2 + 2k + \frac{3}{4}\right) C_1 x + \sum_{K=2}^{\infty} \left(K^2 + k^2 + 2K - \frac{1}{4} \right) C_K x^K \right.$$

$$\left. + \sum_{K=2}^{\infty} C_{K-2} x^K \right] = 0$$

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$$\Rightarrow x^r\left[(r^2-\tfrac{1}{4})c_0+\left((1+r)^2-\tfrac{1}{4}\right)c_1x+\sum_{k=2}^{\infty}\left\{(k+r)^2-\tfrac{1}{4}\right)c_k + c_{k-2}\right\}x^k\Big]x^r=0$$

$$\Rightarrow x^r\left[(r^2-\tfrac{1}{4})c_0+(r^2+2r+\tfrac{3}{4})c_1x+\sum_{k=2}^{\infty}\left\{(k+r)^2-\tfrac{1}{4}\right)c_k + c_{k-2}\right\}x^k\Big]x^r=0$$

using identity properly, we get,

$$(r^2-\tfrac{1}{4})c_0=0 \;;\; (r^2+2r+\tfrac{3}{4})c_1=0$$

$$\left((k+r)^2-\tfrac{1}{4}\right)c_k=-c_{k-2}$$

Since for $c_0=0$ we proceed to a trivial solution which is not our target so $c_0\neq 0$.

$r^2-1/4=0 \Rightarrow r=\pm 1/2$

(Say) $r=r_1=1/2$ $\quad\therefore c_1=0$

$$c_k=\frac{-c_{k-2}}{k(k+2)} \;;\; k=2,3,4,\ldots$$

$$k=2\Rightarrow c_2=\frac{-c_0}{2\cdot 4}\;;\; c_3=\frac{-c_1}{3(5)}=0$$

$$c_4=\frac{-c_2}{4\cdot 6}=\frac{+c_0}{2\cdot 4^2\cdot 6}\;;\; c_5=0$$

and so on

(Say) $r=r_2=-1/2$

$$c_k=\frac{-c_{k-2}}{k(k-2)} \;;\; k=2,3,4,\ldots$$

Hence $y=c_1y_1(x)+c_2y_2(x)$.

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Solve the following Questions.

① $2x^2y'' - xy' + (x^2+1)y = 0$

② $x^2y'' + xy' + (x^2 - \frac{4}{9})y = 0$

③ $xy'' + 2y' - xy = 0$

④ Bessel's Functions.

The equation of the form $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ is called Bessel's equation of order ν.

Series solution of Bessel's equation.

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The Bessel's equation is given as

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad \text{——} \quad (1)$$

Clearly $x = 0$ is a regular singular pt of (1). Now we find the solution of (1) of the form

$$y = \sum_{n=0}^{\infty} C_n x^{n+s} \quad \text{——} \quad (2)$$

$$(1) \Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1) C_n x^{n+s} + \sum_{n=0}^{\infty} (n+s) C_n x^{n+s} + \sum_{n=0}^{\infty} C_n x^{n+s+2} - \nu^2 \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+s)(n+s-1+1) C_n x^{n+s} + \sum_{n=0}^{\infty} C_n x^{n+s+2} - \nu^2 \sum_{n=0}^{\infty} C_n x^{n+s} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \{(n+s)^2 - \nu^2\} C_n x^{n+s} + \sum_{n=0}^{\infty} C_n x^{n+s+2} = 0$$

$$\Rightarrow x^s \left[\sum_{n=0}^{\infty} \{(n+s)^2 - \nu^2\} C_n x^n + \sum_{n=0}^{\infty} C_n x^{n+2} \right] = 0$$

$$\Rightarrow x^s \left[(s^2 - \nu^2) C_0 x^0 + ((1+s)^2 - \nu^2) C_1 x + \sum_{n=2}^{\infty} \{(n+s)^2 - \nu^2\} C_n x^n + \sum_{n=0}^{\infty} C_n x^{n+2} \right] = 0$$

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$$\Rightarrow x^r\left[(r^2-\nu^2)c_0+((1+r)^2-\nu^2)c_1x+\sum_{k=2}^{\infty}\{(k+r)^2-\nu^2\}c_kx^k+\sum_{k=2}^{\infty}c_{k-2}x^k\right]=0$$

$$\Rightarrow x^r\left[(r^2-\nu^2)c_0+((1+r)^2-\nu^2)c_1x+\sum_{k=2}^{\infty}\{((k+r)^2-\nu^2)c_k+c_{k-2}\}x^k\right]=0$$

using identity property, we get,

$(r^2-\nu^2)c_0=0$; $((1+r)^2-\nu^2)c_1=0$

$((k+r)^2-\nu^2)c_k+c_{k-2}=0$

Since for $c_0=0$, we proceed to a trivial solution which is not our target so $c_0\neq 0$.

$\Rightarrow r^2-\nu^2=0$; $c_1=0$

$r=\pm\nu$

(say) $r=r_1=\nu$ so

$$c_k=\frac{-c_{k-2}}{k(k+2\nu)} \quad ; k=2,3,\ldots \longrightarrow (3)$$

(say) $r=r_2=-\nu$ so

$$c_k=\frac{-c_{k-2}}{k(k-2\nu)} \quad ; k=2,3,\ldots \longrightarrow (4)$$

From (3),

$$c_2=\frac{-c_0}{2^2(1+\nu)} \quad ; \quad c_3=0 \quad \because c_1=0$$

$$c_4=+\frac{c_0}{2^4\cdot 1\cdot 2(1+\nu)(2+\nu)} \quad ; \quad c_5=0$$

$$c_6=\frac{c_0}{2^6\cdot 1\cdot 2\cdot 3(1+\nu)(2+\nu)(3+\nu)} \quad ; \quad c_7=0$$

and so on.

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In general,

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n} n!\,(1+\nu)(2+\nu)\cdots(n+\nu)} \quad ; n=1,2,3,\cdots$$

By convention we choose

$$C_0 = \frac{1}{2^\nu}\cdot\frac{1}{\nu!}$$ where ν is an integer.

Thus

$$C_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!\,\nu!\,(1+\nu)(2+\nu)\cdots(n+\nu)}$$

$$C_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!\,(n+\nu)!} \quad ; n=0,1,2\cdots$$

$$\Rightarrow C_{2n} = \frac{(-1)^n}{2^{2n+\nu} n!\,\Gamma(1+\nu+n)} \qquad \because \Gamma(n)=(n-1)!$$

② $\Rightarrow y = \sum_{k=0}^{\infty} C_k x^{k+s} = \sum_{n=0} C_{2n} x^{2n+\nu}$

$$\Rightarrow y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} = J_\nu(x)$$

Similarly for $s = s_2 = -\nu$.

$$y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu} = J_{-\nu}(x).$$

The general solution of ① is

$$y = c_1 y_1(x) + c_2 y_2(x).$$

$$y = c_1 J_{+\nu}(x) + c_2 J_{-\nu}(x)$$

The functions $J_\nu(x)$ and $J_{-\nu}(x)$ are called Bessel's functions of 1st kind of order ν and $-\nu$ respectively.

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Bessel's Function of 1st kind.

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

and $$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}.$$

Where $\nu \neq$ integer.

Bessel's function of 2nd kind.

If $\nu =$ integer, the function defined by

$$Y_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \quad \text{---} (1)$$

and $J_\nu(x)$ are linearly independent solutions of Bessel's equation.

Thus general solution is

$$y = C_1 J_\nu(x) + C_2 Y_\nu(x).$$

As $\nu \to m$, m is an integer, (1) has the indeterminate form $\left(\frac{0}{0}\right)$. However it can be solved by L'Hospital rule as

$$\lim_{\nu\to m} Y_\nu = Y_m = \lim_{\nu\to m} \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$$

Properties of Bessel's functions.

(1) $$xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x).$$

We know that

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

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Diff. w.r.t. 'x', we get

$$J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)\, x^{2n+\nu-1}}{n!\,\Gamma(1+\nu+n)\cdot 2^{2n+\nu}}$$

Multiplying by 'x', we get

$$xJ_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= 2\sum_{n=0}^{\infty} \frac{(-1)^n n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= x\sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} + \nu J_\nu(x)$$

$$= x\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\,\Gamma(1+\nu+k+1)} \left(\frac{x}{2}\right)^{2k+2+\nu-1} + \nu J_\nu(x)$$

$$= x\sum_{k=0}^{\infty} \frac{(-1)'(-1)^k}{k!\,\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+(\nu+1)} + \nu J_\nu(x).$$

$$= -x\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+(\nu+1)} + \nu J_\nu(x)$$

$$= -xJ_{\nu+1}(x) + \nu J_\nu(x)$$

$$\Rightarrow xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x).$$

(2) $$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x).$$

Above result can be written as

$$J_\nu'(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x).$$

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$$\Rightarrow J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x)$$

Multiplying both sides by $x^{-\nu}$, we get

$$x^{-\nu} J_\nu'(x) - x^{-\nu-1} \nu J_\nu(x) = -x^{-\nu} J_{\nu+1}(x)$$

or

$$x^{-\nu} J_\nu'(x) - \nu x^{-\nu-1} J_\nu(x) = -x^{-\nu} J_{\nu+1}(x)$$

or $$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x).$$

(3) $$x J_\nu'(x) = -\nu J_\nu(x) + x J_{\nu-1}(x).$$

We know that

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$\Rightarrow J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)\, x^{2n+\nu-1}}{n!\,\Gamma(1+\nu+n)\, 2^{2n+\nu}}$$

$$\Rightarrow x J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

Putting $2n+\nu = 2(n+\nu) - \nu$, we get

$$x J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+\nu) - \nu}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} - \nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= x J_{\nu-1}(x) - \nu J_\nu(x).$$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+\nu)!} \left(\frac{x}{2}\right)^{2n+\nu} - \nu J_\nu(x) \quad \because \Gamma(n) = (n-1)!$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+\nu-1)!} \frac{x^{2n+\nu}}{2^{2n+\nu-1}} - \nu J_\nu(x).$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+(\nu-1)+n)} \left(\frac{x}{2}\right)^{2n+(\nu-1)} - \nu J_\nu(x)$$

$$= x J_{\nu-1}(x) - \nu J_\nu(x)$$

④ $\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x).$

Above relation can be written as

$$J_\nu'(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x)$$

$$\Rightarrow J_\nu'(x) + \frac{\nu}{x} J_\nu(x) = J_{\nu-1}(x)$$

Multiplying both sides by x^ν, we get,

$$x^\nu J_\nu'(x) + \nu x^{\nu-1} J_\nu(x) = x^\nu J_{\nu-1}(x)$$

$$\Rightarrow \frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x).$$

⑤ $2J_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x).$

Adding Prop. ① and ③, we get,

$$x J_\nu'(x) + x J_\nu'(x) = \nu J_\nu(x) - x J_{\nu+1}(x) + x J_{\nu-1}(x) - \nu J_\nu(x)$$

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$$\Rightarrow 2x J_\nu'(x) = x\left(J_{\nu-1}(x) - J_{\nu+1}(x)\right)$$

$$\Rightarrow J_\nu'(x) = \frac{1}{2}\left[J_{\nu-1}(x) - J_{\nu+1}(x)\right].$$

or

$$2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x).$$

⑥ 2014 $$2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x)$$

Subtracting Prop ③ from ①, we get,

$$xJ_\nu'(x) - xJ_\nu'(x) = \nu J_\nu(x) - xJ_{\nu+1}(x) + \nu J_\nu(x) - xJ_{\nu-1}(x)$$

$$\Rightarrow 0 = 2\nu J_\nu(x) - xJ_{\nu+1}(x) - xJ_{\nu-1}(x)$$

$$\Rightarrow 2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x).$$

or

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x).$$

⑦ $$2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x).$$

We know that

$$J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\,\Gamma(1+\nu+n)} \frac{x^{2n+\nu-1}}{2^{2n+\nu}}$$

Multiplying both sides by 2, we get,

$$2J_\nu'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+\nu)}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

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$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-1)!\,n!\,\Gamma(1+\nu+n)} \frac{x^{2n+\nu-1}}{2^{2n+\nu-2}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\nu+n)!} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(\nu+n-1)!} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\,\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2n+(\nu+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu-1+n)} \left(\frac{x}{2}\right)^{2n+\nu-1}$$

$$= -J_{\nu+1}(x) + J_{\nu-1}(x).$$

$$\Rightarrow 2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x).$$

(8) $2J_0''(x) = J_2(x) - J_1(x)$. (self proceed)

Spherical Bessel Functions.

When the order ν is half an odd integer, that is, $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$ the Bessel function of the 1st kind $J_\nu(x)$ can be expressed in terms of the elementry functions $\sin x$, $\cos x$ and powers of x. Such Bessel functions are called Spherical Bessel function.

(1) $J_0(0) = 1$.

Putting $x=0$, $\nu=0$ in $J_\nu(x)$, we get

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

$$= 1 - (x/2)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \cdots$$

$$J_0(0) = 1.$$

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(2) $J_{\frac{1}{2}}^2(x) + J_{-\frac{1}{2}}^2(x) = \frac{2}{\pi x}$.

We know that

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \quad \text{— (1)}$$

Put $\nu = 1/2$

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}} \quad \text{— (2)}$$

Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

and $\Gamma(1+\alpha) = \alpha\Gamma(\alpha)$.

Therefore,

$n=0 \Rightarrow \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} = \Gamma(3/2)$

$n=1 \Rightarrow \Gamma(1+\frac{1}{2}+1) = \Gamma(5/2) = \frac{1\cdot 3}{2}\Gamma(3/2) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$

$\Rightarrow \Gamma(5/2) = \frac{1\cdot 3}{2^2}\sqrt{\pi}$

$n=2 \Rightarrow \Gamma(1+\frac{1}{2}+2) = \Gamma(1+5/2) = \frac{5}{2}\Gamma(5/2)$

$\Rightarrow \Gamma(\frac{7}{2}) = \frac{5}{2}\cdot\frac{3}{2^2}\sqrt{\pi} = \frac{1\cdot 3\cdot 5}{2^3}\sqrt{\pi}$

and so on.

In general.

$$\Gamma(1+\tfrac{1}{2}+n) = \frac{1\cdot 3\cdot 5\cdots}{2^{n+1}}\sqrt{\pi}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots}{2^{n+1}\cdot 2\cdot n!}\sqrt{\pi}$$

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$$= \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}\cdot n!}$$

Now equation (2) becomes.

$$J_{\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(\frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!}\right)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1} \left(\frac{x}{2}\right)^{-1/2}$$

$$= \left(\frac{2}{x}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\sqrt{\pi}} x^{2n+1}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \longrightarrow (3)$$

Putting $\nu = -\frac{1}{2}$ in (1), we get,

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(1-\frac{1}{2}+n\right)} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}} \longrightarrow (4)$$

$n=0 \Rightarrow \Gamma\left(1-\frac{1}{2}\right) = \Gamma\left(+\frac{1}{2}\right) = \sqrt{\pi}$

$n=1 \Rightarrow \Gamma\left(1-\frac{1}{2}+1\right) = \Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$n=2 \Rightarrow \Gamma\left(\frac{5}{2}\right) = \frac{2\cdot 3}{4}\sqrt{\pi}$; $\Gamma\left(\frac{7}{2}\right) = \frac{1\cdot 3\cdot 5}{8}\sqrt{\pi}$

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and so on.

In general.

$$\Gamma\left(1-\frac{1}{2}+n\right) = \frac{2n!}{2^{2n}\, n!}\sqrt{\pi}$$

Now equation (4) becomes.

$$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\left(\frac{2n!}{2^{2n}n!}\right)\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{2n!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2n} \cdot \left(\frac{x}{2}\right)^{-1/2}$$

$$= \left(\frac{2}{x}\right)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!\sqrt{\pi}} x^{2n}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x \longrightarrow (5)$$

Squaring (3) and (5), then adding, we have,

$$J_{\frac{1}{2}}^2(x) + J_{-\frac{1}{2}}^2(x) = \frac{2}{\pi x}\sin^2 x + \frac{2}{\pi x}\cos^2 x$$

$$\Rightarrow J_{\frac{1}{2}}^2(x) + J_{-\frac{1}{2}}^2(x) = 2/\pi x$$

as required

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(3) $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\cos x - \frac{\sin x}{x}\right)$.

$$J_{\frac{3}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\frac{3}{2}+n)}\left(\frac{x}{2}\right)^{2n+\frac{3}{2}}$$

$$= \frac{x^{3/2}}{2^{3/2}\cdot\Gamma(5/2)} - \frac{x^{7/2}}{\Gamma(7/2)\cdot 2^{7/2}} + \frac{x^{11/2}}{2!\,\Gamma(\frac{11}{2})\,2^{11/2}}$$

$$- \frac{x^{\frac{15}{2}}}{3!\,\Gamma(\frac{15}{2})\,2^{\frac{15}{2}}} + \cdots$$

$$= \frac{x^{3/2}}{2^{3/2}}\left[\frac{1}{\Gamma(5/2)} - \frac{x^2}{\Gamma(\frac{7}{2})\cdot 2^2} + \frac{x^4}{2!\,\Gamma(\frac{9}{2})\,2^4} - \frac{x^6}{3!\,\Gamma(\frac{11}{2})\,2^6}\right.$$

$$\left. + \cdots\right]$$

$$\Gamma(\tfrac{5}{2}) = \frac{3!}{4\cdot 2}\sqrt{\pi} \quad ; \quad \Gamma(\tfrac{7}{2}) = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma(\tfrac{9}{2}) = \frac{7!}{2^7 3!}\sqrt{\pi} \; ; \; \Gamma(\tfrac{11}{2}) = \frac{9!}{2^9 4!}\sqrt{\pi}$$

and so on.

$$J_{\frac{3}{2}}(x) = \frac{x^{3/2}}{2^{3/2}}\left[\frac{1}{\frac{3}{4}\sqrt{\pi}} - \frac{x^2}{\frac{5\times 4}{2^2}\sqrt{\pi}} + \frac{x^4}{2!\,\frac{7\times 6\times 5\times 4}{2^7}\cdot 2^4\sqrt{\pi}}\right.$$

$$\left. - \frac{x^6}{3!\,\frac{9\times 8\times 7\times 6\times 5}{2^9}\sqrt{\pi}\,2^6} + \cdots\right]$$

$$= \frac{x^{3/2}}{2^{3/2}}\left[\frac{4}{3\sqrt{\pi}} - \frac{x^2\,2^3}{2\cdot 5\times 4\sqrt{\pi}} + \frac{x^4\,2^3}{2!\,7\times 6\times 5\times 4\sqrt{\pi}}\right.$$

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$$- \frac{2^3 x^6}{3!\,9\cdot 8\cdot 7\cdot 6\cdot 5\sqrt{\pi}} + \cdots \Big]$$

$$= \frac{2^{1/2}}{\sqrt{\pi x}} \left(\frac{2x^2}{3!} - \frac{4x^4}{5!} + \frac{6x^6}{7!} - \frac{8x^8}{9!} + \cdots \right)$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{(3-1)x^2}{3!} - \frac{(5-1)x^4}{5!} + \frac{(7-1)x^6}{7!} - \frac{(9-1)x^8}{9!} + \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left[-1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots \right]$$

$$+ \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{3!} - \frac{x^4}{5!} + \frac{x^6}{7!} - \frac{x^8}{9!} + \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cdot \frac{1}{x} \left[x - \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{9!} - \cdots \right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}} \frac{1}{x} \sin x - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\cos x - \frac{\sin x}{x} \right).$$

Similarly solve

④ $x J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(-\frac{\sin x}{x} - \cos x \right)$

⑤ $J_{5/2} = ?$

⑥ $J_{-\frac{5}{2}} = ?$

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Example 1: 2017,

Solve $x^2y'' + xy' + (x^2 - 1/4)y = 0$.

Solution.

Given that $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ ——— (1)

The general Bessel's equation is

$x^2y'' + xy' + (x^2 - \nu^2)y = 0$ ——— (2)

Comparing (1) and (2), we get,

$\nu = \pm 1/2 \Rightarrow \nu = 1/2, \frac{-1}{2}$ (not an integer)

Now we find the general solution as

$y = C_1 J_{1/2} + C_2 J_{\frac{1}{2}}(x)$

where $J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n+\frac{1}{2}}$

and

$J_{-\frac{1}{2}}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1-\frac{1}{2}+n)} \left(\frac{x}{2}\right)^{2n-\frac{1}{2}}$.

Example 2: 2012,

Solve $x^2y'' + xy' + (x^2 - 9)y = 0$

Solution.

Given that $x^2y'' + xy' + (x^2 - 9)y = 0$ ——— (1)

Therefore $\nu^2 = 9 \Rightarrow \nu = \pm 3$

which are integers.

The general solution of (1) is

$y = C_1 J_\nu(x) + C_2 Y_\nu(x)$ ——— (2)

where

$J_3(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+3+n)} \left(\frac{x}{2}\right)^{2n+3}$

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and $Y_3(x) = \lim_{\nu \to 3} \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$ $\left(\frac{0}{0}\right)$

By using L'Hospital rule, we get,

$$Y_3(x) = \lim_{\nu \to 3} \frac{-\pi \sin \nu\pi J_\nu(x) + \cos \nu\pi J_\nu'(x) - J_{-\nu}'(x)}{\pi \cos \nu\pi}$$

$$= \frac{-\pi(0) J_3(x) + (-1) J_3'(x) - J_{-3}'(x)}{\pi(-1)}$$

$$\Rightarrow Y_3(x) = \frac{1}{\pi}\left[J_3'(x) + J_3'(x)\right]$$

Now (1) can be written as

$$y = C_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(4+n)} \left(\frac{x}{2}\right)^{2n+3} + \frac{C_2}{\pi}\left(J_3'(x) + J_{-3}'(x)\right)$$

The equation of the form

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0$$

Can be solved by using $t = \alpha x$, $\alpha > 0$.

By chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \alpha\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

Question.

Solve $x^2y'' + xy' + (9x^2 - 4)y = 0$

Solution.

Given that $x^2y'' + xy' + (9x^2 - 4)y = 0$ — (1)

Take $t = 3x$ then $\frac{dt}{dx} = 3$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = 3\frac{dy}{dt}$$

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and $y'' = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = 9\frac{d^2y}{dt^2}$

$① \Rightarrow \frac{t^2}{9}\cdot 9\frac{d^2y}{dt^2} + \frac{t}{3}\cdot 3\frac{dy}{dt} + (t^2-4)y = 0$

$\Rightarrow t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2-4)y = 0 \longrightarrow ②$

$\Rightarrow t^2y'' + ty' + (t^2-4)y = 0 \; ; \; y' = dy/dt$

Comparing with general Bessel's equation we get,

$\nu = 2$ which is an integer.

The general solution of ② is given by

$y = c_1 J_\nu(t) + c_2 Y_\nu(t)$

where

$Y_2(t) = \lim_{\nu\to 2} \frac{\cos\nu\pi\, J_\nu(t) - J_{-\nu}(t)}{\sin\nu\pi}$ $\left(\frac{0}{0}\right)$ form

By using L'Hospital rule,

$Y_2(t) = \lim_{\nu\to 2} \frac{-\pi\sin\nu\pi\, J_\nu(t) + \cos\nu\pi\, J_\nu'(t) - J_{-\nu}'(t)}{\pi\cos\nu\pi}$

$= 0 + \frac{\cos 2\pi\, J_2'(t) - J_{-2}'(t)}{\pi\cos 2\pi}$

$= \frac{J_2'(t) - J_{-2}'(t)}{\pi}$

$Y_2(t) = \frac{1}{\pi}\left(J_2'(t) - J_{-2}'(t)\right)$

Find $J_2'(t)$ and $J_{-2}'(t)$ then by back subsitution get the ans.

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The general solution of an equation of the form

$$y'' + \frac{1-2a}{x} y' + \left(b^2c^2 x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right) y = 0$$

where $p \geq 0$.

is given by

$$y = x^a [C_1 J_p(bx^c) - C_2 Y_p(bx^c)]$$

→ p an integer

→ $J_{-p}(bx^c)$: p not an integer.

Example.

Solve $xy'' + 3y' + 9y = 0$.

Solution.

Given that $xy'' + 3y' + 9y = 0$

Therefore $y'' + \frac{3}{x} y' + \frac{9}{x} y = 0$ ——— (1)

Comparing with general equation, we get,

$1 - 2a = 3$; $b^2c^2 = 9$; $2c - 2 = -1$

and $a^2 - p^2c^2 = 0$

$\Rightarrow a = -1$, $b = 6$, $c = 1/2$, $p = 2$

Thus general solution is

$$y = x^{-1} [C_1 J_2(6x^{1/2}) + C_2 Y_2(6x^{1/2})].$$

Question:

$x^2y'' + (x^2 - 2) y = 0$

Solution.

Given that $x^2y'' + (x^2 - 2) y = 0$ ——— (1)

By Camparison, we get,

$1 - 2a = 0$, $b^2c^2 = 1$, $2c - 2 = 0$

$a^2 - p^2c^2 = -2$

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$a = 1/2$; $b = 1$; $c = 1$, $p = 3/2$

The solution of (1) is

$y = x^{1/2}[C_1 J_{3/2}(x) + C_2 J_{-3/2}(x)]$ ($p \neq$ integer)

Exercise:

Solve the following questions upto 20.

5.3.1 Bessel Functions

In Problems 1–6, use (1) to find the general solution of the given differential equation on $(0, \infty)$.

1. $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$
2. $x^2y'' + xy' + (x^2 - 1)y = 0$
3. $4x^2y'' + 4xy' + (4x^2 - 25)y = 0$
4. $16x^2y'' + 16xy' + (16x^2 - 1)y = 0$
5. $xy'' + y' + xy = 0$
6. $\frac{d}{dx}[xy'] + \left(x - \frac{4}{x}\right)y = 0$

In Problems 7–10, use (12) to find the general solution of the given differential equation on $(0, \infty)$.

7. $x^2y'' + xy' + (9x^2 - 4)y = 0$
8. $x^2y'' + xy' + (36x^2 - \frac{1}{4})y = 0$
9. $x^2y'' + xy' + (25x^2 - \frac{4}{9})y = 0$
10. $x^2y'' + xy' + (2x^2 - 64)y = 0$

In Problems 11 and 12, use the indicated change of variable to find the general solution of the given differential equation on $(0, \infty)$.

11. $x^2y'' + 2xy' + \alpha^2x^2y = 0$; $y = x^{-1/2}u(x)$
12. $x^2y'' + (\alpha^2x^2 - \nu^2 + \frac{1}{4})y = 0$; $y = \sqrt{x}u(x)$

In Problems 13–20, use (18) to find the general solution of the given differential equation on $(0, \infty)$.

13. $xy'' + 2y' + 4y = 0$
14. $xy'' + 3y' + xy = 0$
15. $xy'' - y' + xy = 0$
16. $xy'' - 5y' + xy = 0$
17. $x^2y'' + (x^2 - 2)y = 0$
18. $4x^2y'' + (16x^2 + 1)y = 0$
19. $xy'' + 3y' + x^3y = 0$
20. $9x^2y'' + 9xy' + (x^6 - 36)y = 0$
21. Use the series in (7) to verify that $I_\nu(x) = i^{-\nu}J_\nu(ix)$ is a real function.
22. Assume that b in equation (18) can be pure imaginary, that is, $b = \beta i$, $\beta > 0$, $i^2 = -1$. Use this assumption to express the general solution of the given differential equation in terms of the modified Bessel functions I_n and K_n.
    (a) $y'' - x^2y = 0$ (b) $xy'' + y' - 7x^3y = 0$

In Problems 23–26, first use (18) to express the general solution of the given differential equation in terms of Bessel functions. Then use (23) and (24) to express the general solution in terms of elementary functions.

23. $y'' + y = 0$
24. $x^2y'' + 4xy' + (x^2 + 2)y = 0$
25. $16x^2y'' + 32xy' + (x^4 - 12)y = 0$
26. $4x^2y'' - 4xy' + (16x^2 + 3)y = 0$
27. (a) Proceed as in Example 5 to show that
    $$xJ'_\nu(x) = -\nu J_\nu(x) + xJ_{\nu-1}(x).$$
    [*Hint:* Write $2n + \nu = 2(n + \nu) - \nu$.]
    (b) Use the result in part (a) to derive (21).
28. Use the formula obtained in Example 5 along with part (a) of Problem 27 to derive the recurrence relation
    $$2\nu J_\nu(x) = xJ_{\nu+1}(x) + xJ_{\nu-1}(x).$$

Legendre's Functions.

The equation of the form $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ is called Legendre's equation of order n.

Series Solution of Legendre's Equations.

The Legendre's equation is given by

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ——— (1)

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Clearly $x=0$ is an ordinary point of (1).

Subsituting $y=\sum_{j=0}^{\infty} c_j x^j$ in (1), we get,

$$(1-x^2)\sum_{j=2}^{\infty} j(j-1)c_j x^{j-2} - 2x\sum_{j=1}^{\infty} j c_j x^{j-1} + n(n+1)\sum_{j=0}^{\infty} c_j x^j = 0$$

$$\Rightarrow \sum_{j=2}^{\infty} j(j-1)c_j x^{j-2} - \sum_{j=2}^{\infty} j(j-1)c_j x^j - 2\sum_{j=1}^{\infty} j c_j x^j + n(n+1)\sum_{j=0}^{\infty} c_j x^j = 0$$

$$\Rightarrow \sum_{j=2}^{\infty} \{j(j-1)c_j\} x^{j-2} - \sum_{j=2}^{\infty} j(j-1)c_j x^j - 2\sum_{j=1}^{\infty} j c_j x^j + n(n+1)\sum_{j=0}^{\infty} c_j x^j = 0$$

$$\Rightarrow 2c_2 x^0 + 3\cdot 2 c_3 x^1 + \sum_{j=4}^{\infty} j(j-1)c_j x^{j-2} - \sum_{j=2}^{\infty} j(j-1)c_j x^j - 2c_1 x - 2\sum_{j=2}^{\infty} j c_j x^j + n(n+1)c_0 x^0 + n(n+1)c_1 x + \sum_{j=2}^{\infty} n(n+1)c_j x^j = 0$$

$$\Rightarrow (n(n+1)c_0 + 2c_2) + (n(n+1)c_1 - 2c_1 + 6c_3)x + \sum_{j=4}^{\infty} j(j-1)c_j x^{j-2} - \sum_{j=2}^{\infty} j(j-1)c_j x^j - 2\sum_{j=2}^{\infty} j c_j x^j + \sum_{j=2}^{\infty} n(n+1)c_j x^j = 0$$

$$\Rightarrow [n(n+1)c_0 + 2c_2] + [(n-1)(n+2)c_1 + 6c_3]x$$

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$$+\sum_{K=2}^{\infty}(K+2)(K+1)C_{K+2}x^K+\sum_{K=2}^{\infty}K(K-1)C_Kx^K-2\sum_{K=2}^{\infty}KC_Kx^K$$

$$+\sum_{K=2}^{\infty}n(n+1)C_Kx^K=0$$

$$\Rightarrow[n(n+1)C_0+2C_2]+[(n-1)(n+2)C_1+6C_3]x$$

$$+\sum_{K=2}^{\infty}\{(K+2)(K+1)C_{K+2}-K(K-1)C_K-2KC_K+n(n+1)C_K\}x^K=0$$

$$\Rightarrow[n(n+1)C_0+2C_2]+[(n-1)(n+2)C_1+6C_3]x$$

$$+\sum_{K=2}^{\infty}\{(K+2)(K+1)C_{K+2}-(K(K+1)-n(n+1))C_K\}x^K$$

using identity property, we get,

$$n(n+1)C_0+2C_2=0\ ;\ (n-1)(n+2)C_1+6C_3=0$$

$$(K+2)(K+1)C_{K+2}-[K(K+1)-n(n+1)]C_K=0$$

$$\Rightarrow C_2=-\frac{n(n+1)}{2!}C_0\ ;\ C_3=-\frac{(n-1)(n+2)}{3!}C_1$$

$$C_{K+2}=\frac{K(K+1)-n(n+1)}{(K+2)(K+1)}$$

$$=\frac{(K+1+n)(K-n)}{(K+2)(K+1)}$$

$$K(K+1)-n(n+1)$$
$$=K^2+K-n^2-n$$
$$=K^2+K+nK-nK-n^2-n$$
$$=K(K+1+n)-n(K+n+1)$$

$$\Rightarrow C_{K+2}=-\frac{(n-K)(n+K+1)}{(K+2)(K+1)}\quad \text{where } K=2,3,4,\cdots$$

$$K=2\Rightarrow C_4=-\frac{(n-2)(n+3)}{4\cdot 3}C_2=\frac{(n-2)n(n+1)(n+3)}{4!}C_0$$

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$K=3 \Rightarrow C_5 = -\frac{(n-3)(n+4)}{5\cdot 4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$

$K=4 \Rightarrow C_6 = -\frac{(n-4)(n+5)}{6\cdot 5} C_4$

$= -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} C_0$

$K=5 \Rightarrow C_7 = -\frac{(n-5)(n+6)}{7\cdot 6} C_5$

$= -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} C_1$

and so on.

Thus two linearly independent solutions are

$$y_1(x) = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

and

$$y_2(x) = C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 + \cdots\right]$$

Note: If n is an even integer then $y_1(x)$ terminates whereas $y_2(x)$ is an infinite series.

If n is an odd integer then series for $y_2(x)$ terminates whereas $y_1(x)$ is an infinite.

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Rodrigue's Formula (রড্রিগ্স)

Rodrigue's formula for Legendre's polynomials is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \; ; \; n=0,1,2,\cdots$$

where $P_n(x)$ are called Legendre polynomials.

$n=0 \Rightarrow P_0(x) = 1 \; ; \quad P_1(x) = x$

$$P_2(x) = \frac{1}{2}(3x^2-1) \; ; \quad P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x).$$

Generating Function. 2012,

The Legendre Polynomials $P_n(x)$ can be obtained from its generating function $(1-2xt+t^2)^{-1/2}$ by the identity.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x).$$

$$\Rightarrow 1-\frac{1}{2}(-2xt+t^2) + \frac{(-1/2)(-3/2)}{2!}(-2xt+t^2)^2 + \cdots.$$

$$= \sum_{n=0}^{\infty} t^n P_n(x) \qquad \text{by (binominal theorem)}$$

$$\Rightarrow 1+xt-\frac{1}{2}t^2+\frac{3}{8}(4x^2t^2+t^4-4xt^3)+\cdots = \sum_{n=0}^{\infty} t^n P_n(x)$$

$$\Rightarrow P_0(x) = 1 \; ; \; P_1(x) = x$$

$$P_2(x) = -\frac{1}{2}+\frac{3}{2}x^2 = \frac{1}{2}(3x^2-1) \quad \text{and so on.}$$

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Recurrence Relations for Legendre's Polynomials:

① $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

By generating function identity,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \longrightarrow ①$$

Diff w.r.t. 't', we get,

$$\frac{-1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (1-2xt+t)(1-2xt+t^2)^{-1}(x-t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x).$$

$$\Rightarrow (x-t)(1-2xt+t)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

using ①

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x).$$

$$\Rightarrow x\sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$-2x\sum_{n=0}^{\infty} P_n(x) n t^n + \sum_{n=0}^{\infty} P_n(x) n t^{n+1}$$

After relabelling, we have

$$x\sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=1}^{\infty} P_{n-1}(x) t^n = \sum_{n=-1}^{\infty} P_{n+1}(x)(n+1) t^n$$

$$-2x\sum_{n=0}^{\infty} P_n(x) n t^n + \sum_{n=1}^{\infty} P_{n-1}(x)(n-1) t^n$$

Comparing the coefficients of t^n, $n \geq 1$ on

both sides, we have,

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow x P_n(x) + 2nx P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x) - P_{n-1}(x) + P_{n-1}(x)$$

$$\Rightarrow (1+2n) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$

$$\Rightarrow (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x).$$

(2) $n P_n(x) = x P_n'(x) - P_{n-1}'(x)$.

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \longrightarrow (1)$$

Diff. (1) w.r.t. 't', we get,

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} n t^{n-1} P_n(x) \longrightarrow (2)$$

Diff (1) w.r.t. 'x', we have

$$\frac{-1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} t^n P_n'(x)$$

$$\Rightarrow t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P_n'(x) \longrightarrow (3)$$

Dividing (2) by (3), we have,

$$\frac{x-t}{t} = \frac{\sum_{n=1}^{\infty} n t^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P_n'(x)}$$

$$\Rightarrow x \sum_{n=0}^{\infty} P_n'(x) t^n - \sum_{n=0}^{\infty} P_n'(x) t^{n+1} = \sum_{n=1}^{\infty} P_n(x) n t^n$$

After relabelling, we have,

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$$\Rightarrow x\sum_{n=0}^{\infty} P_n'(x)t^n - \sum_{n=1}^{\infty} P_{n-1}'(x)t^n = \sum_{n=1}^{\infty} P_n(x)\,nt^n$$

Comparing the coefficients of $t^n, t^n \geq 1$, on both sides, we have

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x)$$

or $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

(3) $(n+1)P_{n+1}(x) = xP_{n+1}'(x) - P_n'(x)$

By replacing 'n' by 'n+1' in Prop (2), we get

$$(n+1)P_n(x) = xP_{n+1}'(x) - P_n'(x).$$

(4) $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Diff. w.r.t. 'x', we get,

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=1}^{\infty} P_n'(x)t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)t^n$$

$$\Rightarrow t\sum_{n=0}^{\infty} t^n P_n(x) = \sum_{n=1}^{\infty} P_n'(x)t^n - 2x\sum_{n=1}^{\infty} P_n'(x)t^{n+1} + \sum_{n=1}^{\infty} P_n'(x)t^{n+2}$$

$$\Rightarrow \sum_{n=0}^{\infty} P_n(x)t^{n+1} = \sum_{n=0}^{\infty} P_{n+1}'(x)t^{n+1} - 2x\sum_{n=1}^{\infty} P_n'(x)t^{n+1} + \sum_{n=2}^{\infty} P_{n-1}'(x)t^{n+1}$$

Comparing the coefficents of t^{n+1}, we get,

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$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x) \quad \text{---(1)}$$

The Prop (1) is given as

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \quad \text{---(2)}$$

Dif. (2), w.r.t x, we get,

$$(n+1)P'_{n+1}(x) = (2n+1)[P_n(x) + xP'_n(x)] - nP'_{n-1}(x)$$

$$\Rightarrow (n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1} \quad \longrightarrow (3)$$

$$(1) \Rightarrow P'_{n-1}(x) = P_n(x) - P'_{n+1}(x) + 2xP'_n(x) \quad \text{---(4)}$$

Subsituting (4) in (3), we get,

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - n[P_n(x) - P'_{n+1}(x) + 2xP'_n(x)]$$

$$\Rightarrow nP'_{n+1}(x) + P'_{n+1}(x) = 2nP_n(x) + P_n(x) + 2nxP'_n(x) + xP'_n(x) - nP_n(x) + nP'_{n+1}(x) - 2nxP'_n(x)$$

$$\Rightarrow P'_{n+1}(x) = P_n(x) + xP'_n(x) + nP'_n(x)$$

$$\Rightarrow P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)$$

or $P'_{n+1}(x) - xP'_n(x) = (n+1)P_n(x)$.

Orthogonal Polynomials.

If $\varphi_n(x)$ is a set of polynomials in x, of each degree, then $\varphi_n(x)$ is said to be orthogonal if there exists an interval $[a,b]$ and a weight function $w(x) > 0$, $\forall\, x \in [a,b]$ such that $\int_a^b \varphi_n(x)\,\varphi_m(x)\,w(x)\,dx = 0$

where $m \neq n$.

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Question.

Prove that Legendre Polynomials $P_n(x)$ are orthogonals w.r.t. weight function $w(x)=1$ over the interval $[-1, 1]$.

Solution.

The Legendre equation of order n is given by

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{——} (1)$$

or $$\frac{d}{dx}[(1-x^2)y'] + n(n+1)y = 0 \quad \text{——} (2)$$

Since $P_n(x)$ satisfies the DE (2), Therefore,

$$\frac{d}{dx}[(1-x^2)P_n'(x)] + n(n+1)P_n(x) = 0 \longrightarrow (3)$$

Multiplying (3) by $P_m(x)$ and then integrating w.r.t. x from -1 to $+1$, we have,

$$\int_{-1}^{1} P_m(x)\frac{d}{dx}[(1-x^2)P_n'(x)]\,dx + n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

Integrating, 1st factor, by parts, we get.

$$P_m(x)(1-x^2)P_n'(x)\Big|_{-1}^{1} - \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx + n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0$$

$$\Rightarrow -\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx + n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx \longrightarrow (4)$$

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Interchanging the role of m and n in (4), we get,

$$-\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx+n(n+1)\int_{-1}^{1}P_m(x)P_n(x)dx=0 \longrightarrow (5)$$

, Subtracting (5) from (4), we have,

$$-\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)dx+n(n+1)\int_{-1}^{1}P_n(x)P_m(x)dx$$

$$+\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx-m(m+1)\int_{-1}^{1}P_m(x)P_n(x)dx=0$$

$$\Rightarrow [n(n+1)-m(m+1)]\int_{-1}^{1}P_m(x)P_n(x)dx=0$$

Since m and n are non negative, $m\neq n$.

Therefore,

$$\int_{-1}^{1}P_m(x)P_n(x)dx=0$$

which shows the orthogonality of Legendre function over the interval $[-1,1]$.

Orthogonality of Bessel Functions.

The Bessel's DE of order ν is given as

$$x^2y''+xy'+(x^2-\nu^2)y=0 \longrightarrow (1)$$

The slightly different form of (1) in λ parameter is

$$x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0 \longrightarrow (2) \quad \because x=\lambda t$$

The solution of (2) will be $J_\nu(\lambda t)$.

Furthermore, (2) can be written as,

$$x^2y''+xy'+(\lambda^2x^2-\nu^2)y=0 \longrightarrow (3)$$

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For orthogonality of Bessel's functions, we put $J_\nu(\alpha x) \equiv f(x)$ and $J_\nu(\beta x) \equiv g(x)$.

Then they satisfy the equations

$$x^2 f'' + xf' + (\alpha^2 x^2 - \nu^2) f = 0 \quad \text{---} (4)$$

and $$x^2 g'' + xg' + (\beta^2 x^2 - \nu^2) g = 0 \quad \text{---} (5)$$

Xing (4) by g and (5) by f then subtracting we get,

$$x^2 gf'' + xgf' + g(\alpha^2 x^2 - \nu^2) f - x^2 fg'' - xfg' - (\beta^2 x^2 - \nu^2) fg = 0$$

$$\Rightarrow x^2(gf'' - fg'') + x(gf' - fg') + fg(\alpha^2 x^2 - \nu^2 - \beta^2 x^2 + \nu^2) = 0$$

$$\Rightarrow x^2(gf'' - fg'') + x(gf' - fg') + fg(\alpha^2 x^2 - \beta^2 x^2) = 0$$

$$\Rightarrow (gf'' - fg'')x + (gf' - fg') + x(\alpha^2 - \beta^2) fg = 0$$

or $$\frac{d}{dx}[x(gf' - fg')] + x(\alpha^2 - \beta^2) fg = 0$$

Integrating w.r.t x from a to b, we have

$$\int_a^b \frac{d}{dx}[x(gf' - fg')]\,dx + (\alpha^2 - \beta^2)\int_a^b xfg\,dx = 0$$

$$[x(gf' - fg')]_a^b = (\beta^2 - \alpha^2)\int_a^b x f(x) g(x)\,dx$$

$$\Rightarrow \int_a^b x f(x) g(x)\,dx = \frac{1}{\beta^2 - \alpha^2}\left[x(gf' - fg')\right]_a^b$$

Replacing $f(x), g(x)$ by $J_\nu(\alpha x), J_\nu(\beta x)$, we get

$$\int_a^b x J_\nu(\alpha x) J_\nu(\beta x)\,dx = \frac{x}{\beta^2 - \alpha^2}\left[J_\nu'(\alpha x) J_\nu(\beta x) - J_\nu'(\beta x) J_\nu(\alpha x)\right]_a^b$$

If $\alpha \neq \beta$ then R.H.S vanishes at the end points of integral, then $\int_a^b x J_\nu(\alpha x) J_\nu(\beta x)\,dx = 0$

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Example 1:

Show that $P_n(-x) = (-1)^n P_n(x)$.

Solution.

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad \text{——} (1)$$

Replacing x and t simultaneously by $-x$ and $-t$ in (1), we have,

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-x)(-t)^n$$

$$\Rightarrow (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n P_n(-x)\, t^n \quad \text{——} (2)$$

Comparing (1) and (2), we get,

$$\sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} (-1)^n P_n(-x)\, t^n$$

Comparing the coefficients of t^n, we have,

$$P_n(x) = (-1)^n P_n(-x)$$

or

$$P_n(-x) = (-1)^n P_n(x) \quad \text{as required.}$$

Example 2:

Show that $P_{2m+1}(0) = \dfrac{(-1)^m\, 2m!}{2^{2m}\,(m!)^2}$

Solution.

We know that

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \quad \text{——} (1)$$

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Put $x=0$ in (1), then

$$\sum_{n=0}^{\infty} P_n(0)t^n = (1+t^2)^{-1/2}$$

by using Binomial theorem, we have,

$$\sum_{n=0}^{\infty} P_n(0)t^n = 1 - \frac{1}{2}t^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}t^4 - \frac{1}{2}\frac{(-3/2)(-5/2)}{3!}t^6 + \cdots$$

$$= 1 - \frac{1}{2\cdot 1!}t^2 + \frac{1\cdot 3}{2^2\cdot 2!}t^4 - \frac{1\cdot 3\cdot 5}{2^3\,3!}t^6 + \cdots$$

$$= 1 - \frac{1}{2\cdot 1!}t^2 + \frac{1\cdot 3}{2^2\cdot 2!}t^4 - \frac{1\cdot 3\cdot 5}{2^3\,3!}t^6 + \cdots + (-1)^m \frac{1\cdot 3\cdot 5\cdots(2m-1)}{2^m\, m!}t^{2m} + \cdots$$

Comparing the coefficients of $n=2m+1$, we get,

$$P_{2m+1}(0) = \frac{(-1)^m 1\cdot 3\cdot 5\cdots(2m-1)}{2^m\, m!} = \frac{(-1)^m 2m!}{2^m\, m!}$$

Example 3:

Prove that $\int_{-1}^{+1} x^2 P_{n-1}(x) P_{n+1}(x)\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

Solution.

We know the recurrence relation

$$(2n+1)x P_n = (n+1)P_{n+1} + nP_{n-1}$$

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Replacing n by $n+1$ and $n-1$ respectively, we get

$$(2n+3)x P_{n+1} = (n+2) P_{n+2} + (n+1) P_n$$

and

$$(2n-1)x P_{n-1} = n P_n + (n-1) P_{n-2}$$

Xing the corresponding sides of the above equations and integrating w.r.t. x from $x=-1$ to $x=+1$, we have

$$(2n+3)(2n-1)\int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\, dx = n(n+2)\int_{-1}^{1} P_n P_{n+2}\, dx + (n+1)(2n-1)\int_{-1}^{+1} P_{n+2} P_{n-1}\, dx + n(n+1)\int_{-1}^{1} P_n^2\, dx + (n+1)(n-1)\int_{-1}^{1} P_n P_{n-2}\, dx \quad \text{——— (1)}$$

using the orthogonality conditions for the Legendre Polynomials

$$\int_{-1}^{+1} P_n P_m\, dx = \frac{2}{2n+1}\cdot \delta_{mn}$$

, we find that all terms on the R.H.S of (1) are zero except one.

Therefore, we have

$$(2n+3)(2n-1)\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\, dx = n(n+1)\int_{-1}^{1} P_n^2\, dx$$

$$= n(n+1) \times \frac{2}{2n+1} = \frac{2n(n+1)}{2n+1}$$

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CHAP 2:

Hypergeometric Function.

Pochhammer Symbol:

Let n be a +ive integer. Then Pochhammer Symbol is defined by

$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ with $(\alpha)_0 = 1$

Properties.

① $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

L.H.S ⇒

$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$

$= \dfrac{1\cdot2\cdot3\cdots(\alpha-1)\,\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)}{1\cdot2\cdot3\cdots(\alpha-1)}$

$= \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = R.H.S \because \Gamma(P) = (P-1)!$

② $(\alpha)_{n+1} = \alpha(\alpha+1)_n$

$L.H.S \Rightarrow (\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+(n+1)-1)$

$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n+1-1)$

$= \alpha(\alpha+1)(\alpha+2)\cdots((\alpha+1)+(n-1))$

$= \alpha[(\alpha+1)(\alpha+2)\cdots((\alpha+1)+n-1)]$

$= \alpha(\alpha+1)_n = R.H.S.$

③ $(\alpha+n)(\alpha)_n = (\alpha)_{n+1}$

$L.H.S \Rightarrow (\alpha+n)(\alpha)_n = (\alpha+n)[\alpha(\alpha+1)\cdots(\alpha+n-1)]$

$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\alpha+n)$

$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(\alpha+(n+1)-1)$

$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+(n+1)-1)$

$= (\alpha)_{n+1}$

$= R.H.S.$

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2011, 2018, 2016,

General hypergeometric function.

The GHGF is defined by

$${}_mF_n(\alpha_1, \alpha_2, \cdots \alpha_m; \beta_1, \beta_2, \cdots \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r \cdots (\alpha_m)_r}{(\beta_1)_r (\beta_2)_r \cdots (\beta_n)_r} \cdot \frac{x^r}{r!}$$

The general HGF is also denoted by

$${}_mF_n\left[\begin{matrix}\alpha_1, \alpha_2, \cdots, \alpha_m; \\ \beta_1, \beta_2, \cdots, \beta_n;\end{matrix} x\right] = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r \cdots (\alpha_m)_r}{(\beta_1)_r (\beta_2)_r \cdots (\beta_n)_r} \cdot \frac{x^r}{r!}$$

Case 1: For $m = n = 1$.

$${}_1F_1(\alpha; \beta; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \cdot \frac{x^r}{r!}$$

or $F(\alpha; \beta; x) = 1 + \frac{\alpha}{\beta} x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} \frac{x^2}{2!} + \cdots$

This is known as Confluent hypergeometric (or Kummer) function.

Case 2: For $m = 2$, $n = 1$

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!}$$

or

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \cdots \longrightarrow (1)$$

Particular case.

If we take $\alpha = 1$, $\beta = \gamma$, then

$$F(1, \beta; \gamma; x) = 1 + x + x^2 + x^3 + \cdots$$

which is geometric series.

As a particular case (1) is called hypergeometric series.

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Gauss's Hypergeometric Equation.

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0$$

is called Gauss's hypergeometric equation.

Symmetric Property of hypergeometric function.

Hyp. G. F. does not change if the parameters α and β are interchanged, keeping γ fixed.

Thus $F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x)$

Proof.

We know that

$$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \quad \text{---} (1)$$

and $$F(\beta, \alpha; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\beta)_r (\alpha)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \quad \text{---} (2)$$

$$\Rightarrow F(\beta, \alpha; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \quad \text{---} (3)$$

From (1) and (3), we can write.

$$F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x)$$

Differentiation of hypergeometric functions.

Show that $\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; x)$

and deduce that

$$\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha+n, \beta+n; \gamma+n; x) \longrightarrow (1)$$

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Proof.

Since we know that

$$F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!}$$

Diff. both sides w.r.t. x, we get,

$$\frac{d}{dx} F(\alpha,\beta;\gamma;x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{r\, x^{r-1}}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{r\, x^{r-1}}{r(r-1)!}$$

$$= \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^{r-1}}{(r-1)!}$$

Put $m = r-1$, then $r = m+1$

$$\frac{d}{dx} F(\alpha,\beta;\gamma;x) = \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} \cdot \frac{x^m}{m!}$$

By Pochhammer Symbol, (Prop 2)

$$\frac{d}{dx} F(\alpha,\beta;\gamma;x) = \sum_{m=0}^{\infty} \frac{\alpha(\alpha+1)_m \, \beta(\beta+1)_m}{\gamma(\gamma+1)_m} \cdot \frac{x^m}{m!}$$

$$= \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \cdot \frac{x^m}{m!}$$

$$\Rightarrow \frac{d}{dx} F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \cdot \frac{x^m}{m!} \longrightarrow (2)$$

$$\Rightarrow \frac{d}{dx} F(\alpha,\beta;\gamma;x) = \frac{\alpha\beta}{\gamma} F(\alpha+1,\beta+1;\gamma+1;x)$$

i.e. eq (1)

We prove deduction by mathematical induction method on n.

For $n=1$ (1) is true which is given in (2). the result holds

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Suppose ① is true for $n=m$, we show that it is true for $n=m+1$.

Differentiating ①, w.r.t. x, we get

$$\frac{d^{m+1}}{dx^{m+1}} F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{d}{dx} F(\alpha+m,\beta+m;\gamma+m;x)$$

$$= \frac{(\alpha)_m (\beta)_m (\alpha+m)(\beta+m)}{(\gamma)_m (\gamma+m)} F(\alpha+m+1,\beta+m+1;\gamma+m+1;x)$$

$$= \frac{(\alpha+m)(\alpha)_m (\beta)_m (\beta+m)}{(\gamma+m)(\gamma)_m} F(\alpha+m+1,\beta+m+1;\gamma+m+1;x)$$

By Pochhamer Symbol property

$$(\alpha+n)(\alpha)_n = (\alpha)_{n+1}$$

$$\frac{d^{m+1}}{dx^{m+1}} F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}} F(\alpha+m+1,\beta+m+1;\gamma+m+1;x)$$

This shows that ① is true for $n=m+1$.

Hence ① is true for each +ive integer.

Put $x=0$ in ①, we get,

$$\frac{d^n}{dx^n} F(\alpha,\beta;\gamma;x)\Big|_{x=0} = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha+n,\beta+n,\gamma+n,0)$$

$$= \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \left[\sum_{k=0}^{\infty} \frac{(\alpha+n)_k (\beta+n)_k}{(\gamma+n)_k} \cdot \frac{x^k}{k!}\right]_0$$

$$= \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} (1)$$

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$$\Rightarrow \frac{d^n}{dx^n} F(\alpha,\beta;\gamma;x)\Big|_{x=0} = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}$$

Integral Representation for the hypergeometric function.

$$F(\alpha,\beta;\gamma;x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha} dt \quad ; \gamma > \beta > 0$$

or

$$F(\alpha,\beta;\gamma;x) = \frac{1}{B(\beta,\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha} dt \quad ; \gamma > \beta > 0$$

where

B is beta function which is given by

$$B(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$$

B-function in integral form is given by

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$$

Gauss's Theorem.

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

Proof.

We know that

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha} dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1} dt$$

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$\rightarrow \beta(\beta, \gamma-\beta-\alpha)$

Using β-function in integral form, we get.

$$F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta)\cdot\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}$$

$$\Rightarrow F(\alpha,\beta;\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}$$ as required.

Examples:

① Show that $e^x = {}_1F_1(\alpha;\alpha;x)$

Since we know that

$${}_1F_1(\alpha;\beta;x) = \sum_{s=0}^{\infty} \frac{(\alpha)_s}{(\beta)_s} \cdot \frac{x^s}{s!}$$

$$\Rightarrow {}_1F_1(\alpha;\beta;x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots$$

Putting $\beta=\alpha$, we get

$${}_1F_1(\alpha;\alpha;x) = 1 + \frac{\alpha}{\alpha}x + \frac{\alpha(\alpha+1)}{\alpha(\alpha+1)}\frac{x^2}{2!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots$$

$$\Rightarrow {}_1F_1(\alpha;\alpha;x) = e^x$$

② Show that $(1-x)^{-\alpha} = {}_2F_1(\alpha;\beta;\beta;x)$

We know that

$${}_2F_1(\alpha,\beta;\gamma;x) = \sum_{s=0}^{\infty} \frac{(\alpha)_s(\beta)_s}{(\gamma)_s} \cdot \frac{x^s}{s!}$$

$$= 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

Putting $\gamma=\beta$, we get

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$${}_2F_1(\alpha,\beta;\beta;x) = 1 + \frac{\alpha\beta}{\beta}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots$$

$$= 1 + \alpha x + \alpha(\alpha+1)\frac{x^2}{2!} + \cdots$$

$$= 1 + (\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}x^2 + \cdots$$

$$\Rightarrow {}_2F_1(\alpha,\beta;\beta;x) = (1-x)^{-\alpha} \quad \text{(by binomial theorem)}$$

③ Show that $(1-x)^{-1} = F(1,1;1;x)$, $|x|<1$

We know that

$$F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

Putting $\alpha=\beta=\gamma=1$, we get,

$$F(1,1;1;x) = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\Rightarrow F(1,1;1;x) = (1-x)^{-1} \quad \text{(by B. theorem)}$$

④ Show that $\ln(1+x) = x\,F_1(1,1;2;-x)$.

We know that

$${}_2F_1(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

Putting $\alpha=1$, $\beta=1$, $\gamma=2$ and $x=-x$, we get,

$${}_2F_1(1,1;2;-x) = 1 + \frac{1}{2}(-x) + \frac{1(1+1)(1+1)}{2(2+1)}\frac{x^2}{2!} + \cdots$$

$$= 1 - \frac{x}{2} + \frac{4}{6}\frac{x^2}{2!} - \cdots$$

$$= 1 - \frac{x}{2} + \frac{2}{3}\frac{x^2}{2} - \cdots$$

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$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$$

$$\Rightarrow {}_2F_1(1, 1; 2; -x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \cdots$$

Multiplying both sides by 'x', we get

$$x\,{}_2F_1(1, 1; 2; -x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\Rightarrow x\,{}_2F_1(1, 1; 2; -x) = \ln(1+x).$$

(5) Show that $\log\left(\frac{1+x}{1-x}\right) = 2xF\left(\frac{1}{2}, 1; \frac{3}{2}; x^2\right)$

(6) Show that $\sin^{-1}x = xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$

We know that

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$$

Putting $\alpha = 1/2$, $\beta = 1/2$, & $\gamma = 3/2$ and $x = x^2$, then, we get,

$$F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = 1 + \frac{1/4}{3/2}x^2 + \frac{\frac{1}{2}\left(\frac{3}{2}\right)\cdot\frac{1}{2}\left(\frac{3}{2}\right)}{\frac{3}{2}\left(\frac{5}{2}\right)}\frac{x^4}{2!} + \cdots$$

$$= 1 + \frac{1}{6}x^2 + \frac{3}{20}\frac{x^4}{2!} + \cdots$$

$$\Rightarrow F\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = 1 + \frac{1^2 \cdot x^2}{3!} + \frac{1^2 \cdot 3^2 \cdot x^4}{5!} + \cdots$$

Multiplying by x, we get

$$xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = x + \frac{1^2 \cdot x^3}{3!} + \frac{1^2 \cdot 3^2 \cdot x^5}{5!} + \cdots$$

$$\Rightarrow xF\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = \sin^{-1}x$$

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⑦ Show that $\tan^{-1}x = x F\left(\frac{1}{2}, 1; \frac{3}{2}; -x^2\right)$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

⑧ Show that $\lim_{a\to\infty} {}_2F_1\left(1, a; 1; \frac{x}{a}\right) = e^x$

We know that

${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots$

Putting $\alpha = 1$, $\beta = a$, $\gamma = 1$ and $x = x/a$, we get,

${}_2F_1\left(1, a; 1; \frac{x}{a}\right) = 1 + \frac{a}{1}\left(\frac{x}{a}\right) + \frac{2a(a+1)}{2!}\left(\frac{x}{a}\right)^2 + \cdots$

$= 1 + x + \frac{(a+1)}{a}\frac{x^2}{2!} + \cdots$

$\Rightarrow {}_2F_1\left(1, a; 1; \frac{x}{a}\right) = 1 + x + (1 + 1/a)\frac{x^2}{2!} + \cdots$

Taking $\lim_{a\to\infty}$ on both sides, we get

$\lim_{a\to\infty} {}_2F_1\left(1, a; 1; \frac{x}{a}\right) = \lim_{a\to\infty}\left[1 + x + \left(1 + \frac{1}{a}\right)\frac{x^2}{2!} + \cdots\right]$

$= 1 + x + \frac{x^2}{2!} + \cdots$

$\Rightarrow \lim_{a\to\infty} {}_2F_1\left(1, a; 1; \frac{x}{a}\right) = e^x.$

⑨ Show that ${}_2F_1(a; 1; a; x) = (1-x)^{-1}$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

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(10) Show that $\frac{d}{dx}\,{}_2F_1(a,b;c;x)\Big|_{x=0} = \frac{ab}{c}$

We know that

$$\frac{d^n}{dx^n}\,{}_2F_1(\alpha,\beta;\gamma;x)\Big|_{x=0} = \frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n} \quad \text{—— (1)}$$

Subsituting $\alpha=a$; $\beta=b$; $\gamma=c$ and $n=1$, in (1), we get

$$\frac{d}{dx}\,{}_2F_1(a,b,c,x)\Big|_{x=0} = \frac{ab}{c} \qquad \because (\alpha)_0 = 1$$

(11) Show that

$$(1-nx)\,{}_2F_1(1-n,1;2;x) = (1-x)^n$$

where n is any natural numbers.

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \cdots$$

(12) Find the third derivative of ${}_2F_1(2,3;1;x)$ w.r.t. x.

We know that

$$\frac{d^n}{dx^n}F(\alpha,\beta;\gamma;x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n,\beta+n;\gamma+n;x)$$

Subsituting $\alpha=2$, $\beta=3$, $\gamma=1$ and $n=3$, we get

$$\frac{d^3}{dx^3}F(2;3;1;x) = \frac{(\alpha)_3\,(\beta)_3}{(\gamma)_3}F(5,6;4;x) \longrightarrow (1)$$

Since

$$F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}x^2 + \cdots$$

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Subsituting $\alpha=5$, $\beta=6$ and $\gamma=4$, we get

$$F(5,6;4;x) = 1+\frac{5\cdot 6}{4}x+\frac{5(6)6(7)}{4(5)}x^2+\cdots$$

$$\Rightarrow F(5,6;4;x) = 1+\frac{5\cdot 6}{4}x+\frac{7\cdot 6^2\cdot 5}{5\cdot 4}x^2+\cdots \quad \text{---} (2)$$

and $(\alpha)_3 = (2)_3 = 2(2+1)(2+2)$

$$= 2(3)(4) = 24$$

$$(\beta)_3 = (3)_3 = 3(3+1)(3+2)$$

$$= 3(4)(5) = 60$$

$$(\gamma)_3 = (1)_3 = 1(1+1)(1+2)$$

$$= 1(2)(3) = 6$$

so $$\frac{(\alpha)_3(\beta)_3}{(\gamma)_3} = \frac{(24)60}{6} = (4)(60) = 240 \quad \text{---} (3)$$

Subsituting (2) and (3) in (1), we get,

$$\frac{d^3}{dx^3}F(2,3;1;x) = 240\left(1+\frac{5\cdot 6}{4}x+\frac{7\cdot 6^2\cdot 5}{5\cdot 4}x^2+\cdots\right)$$

Question 1.

(a) Prove that following relations holds.

$$F(\alpha-1,\beta-1;\gamma;x)-F(\alpha,\beta-1;\gamma;x)$$

$$= \left(\frac{1-\beta}{\gamma}\right)x\,F(\alpha,\beta;\gamma+1;x).$$

(b) $$\alpha F(\alpha+1,\beta;\gamma;x)-(\gamma-1)F(\alpha,\beta;\gamma-1;x)$$

$$= (\alpha+1-\gamma)F(\alpha,\beta;\gamma;x).$$

Solution.

(a) we know that

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$$F(\alpha,\beta;\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots \longrightarrow (1)$$

Now,

$$F(\alpha-1,\beta-1;\gamma;x) = 1 + \frac{(\alpha-1)(\beta-1)}{\gamma}x + \frac{(\alpha-1)\alpha(\beta-1)\beta}{\gamma(\gamma+1)}\frac{x^2}{2!}$$
$$+ \frac{(\alpha-1)\alpha(\alpha+1)(\beta-1)\beta(\beta+1)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots \longrightarrow (2)$$

and

$$F(\alpha,\beta-1;\gamma;x) = 1 + \frac{\alpha(\beta-1)}{\gamma}x + \frac{\alpha(\alpha+1)(\beta-1)\beta}{\gamma(\gamma+1)}\frac{x^2}{2!}$$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)(\beta-1)\beta(\beta+1)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots \longrightarrow (3)$$

Subtracting (3) from (2), we get,

$$F(\alpha-1,\beta-1;\gamma;x) - F(\alpha,\beta-1;\gamma;x) = 1 - 1$$
$$+ \frac{(\alpha-1)(\beta-1) - \alpha(\beta-1)}{\gamma}x + \frac{(\alpha-1)\alpha(\beta-1)\beta - \alpha(\alpha+1)(\beta-1)\beta}{\gamma(\gamma+1)}\frac{x^2}{2!}$$
$$+ \frac{(\alpha-1)\alpha(\alpha+1)(\beta-1)\beta(\beta+1) - \alpha(\alpha+1)(\alpha+2)(\beta-1)\beta(\beta+1)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3} + \cdots$$
$$= \frac{(1-\beta)}{\gamma}x\left[1 - \alpha + \alpha + \frac{(1-\alpha)\alpha\beta + \alpha(\alpha+1)\beta}{\gamma+1}\frac{x}{2!} + \right.$$
$$\left. \frac{(1-\alpha)\alpha(\alpha+1)\beta(\beta+1) + \alpha(\alpha+1)(\alpha+2)\beta(\beta+1)}{(\gamma+1)(\gamma+2)}\frac{x^2}{3!} + \cdots\right]$$
$$= \frac{(1-\beta)}{\gamma}x\left[1 + \frac{2\alpha\beta}{\gamma+1}\frac{x}{2} + \frac{\alpha(\alpha+1)\beta(\beta+1)(1-\alpha+\alpha+2)}{(\gamma+1)(\gamma+2)}\frac{x^2}{6} + \cdots\right]$$

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$$= \left(\frac{1-\beta}{\gamma}\right) x \left[1 + \frac{\alpha\beta}{\gamma+1} x + \frac{\alpha(\alpha+1)\beta(\alpha+1)}{(\gamma+1)(\gamma+2)} \frac{x^2}{2!} + \cdots\right]$$

$$= \left(\frac{1-\beta}{\gamma}\right) x \; F(\alpha, \beta; \gamma+1; x).$$

(b) From (1),

$$F(\alpha+1, \beta; \gamma; x) = 1 + \frac{(\alpha+1)\beta}{\gamma} x + \frac{(\alpha+1)(\alpha+2)\beta(\beta+1)x^2}{\gamma(\gamma+1)\,2!}$$

$$+ \cdots$$

$$\Rightarrow \alpha F(\alpha+1, \beta; \gamma; x) = \alpha + \frac{\alpha(\alpha+1)\beta}{\gamma} x + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)}{\gamma(\gamma+1)}$$

$$\frac{x^2}{2!} + \cdots \longrightarrow (4)$$

Now, $F(\alpha, \beta; \gamma-1; x) = 1 + \frac{\alpha\beta}{\gamma-1} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{(\gamma-1)(\gamma)} \frac{x^2}{2!}$

$$+ \cdots$$

$$(\gamma-1) F(\alpha, \beta; \gamma-1; x) = \gamma - 1 + \alpha\beta x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma} \frac{x^2}{2!}$$

$$+ \cdots \longrightarrow (5)$$

Subtracting (5) from (4), we get,

$$\alpha F(\alpha+1, \beta; \gamma; x) - (\gamma-1) F(\alpha, \beta; \gamma-1; x) = \alpha - \gamma + 1$$

$$+ \left(\frac{\alpha(\alpha+1)\beta}{\gamma} - \alpha\beta\right) x + \left(\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)}{\gamma(\gamma+1)} - \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma}\right)$$

$$\frac{x^2}{2!} + \cdots$$

$$= \alpha + 1 - \gamma + \frac{\alpha^2\beta + \alpha\beta - \alpha\beta\gamma}{\gamma} x +$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1) - \alpha(\alpha+1)\beta(\beta+1)(\gamma+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \cdots$$

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$$= \alpha+1-\gamma + \frac{\alpha\beta}{\gamma}(\alpha+1-\gamma)x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}(\alpha+2-\gamma-1)\frac{x^2}{2!} + \cdots$$

$$= (\alpha+1-\gamma)\left[1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \cdots\right]$$

$$= (\alpha+1-\gamma)\,F(\alpha,\beta;\gamma;x).$$

Question 2:

Prove that $F(\alpha,\beta;\gamma;\frac{1}{2}) = 2^{\alpha}F(\alpha,\gamma-\beta;\gamma;-1)$

Solution.

We know that

$$F(\alpha,\beta;\gamma;x) = (1-x)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right)$$

Put $x=\frac{1}{2}$ in above equation,

$$F(\alpha,\beta;\gamma;\tfrac{1}{2}) = (1-\tfrac{1}{2})^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{\frac{1}{2}}{-\frac{1}{2}}\right)$$

$$= (\tfrac{1}{2})^{-\alpha}F(\alpha,\gamma-\beta;\gamma;-1)$$

$$= \frac{1}{(2)^{-\alpha}}F(\alpha,\gamma-\beta;\gamma;-1)$$

$$\Rightarrow F(\alpha,\beta;\gamma;\tfrac{1}{2}) = 2^{\alpha}F(\alpha,\gamma-\beta;\gamma;-1).$$

Question 3:

(a) $\left(1+\frac{x}{\alpha}\right)e^{x} = F(\alpha+1;\alpha;x)$

(b) $e^{x}-1 = xF(1;2;x)$ · (self).

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Solution.

(a) We know that

$$F(\alpha;\beta;x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots$$

Replacing α by $\alpha+1$ and β by α, we have,

$$F(\alpha+1;\alpha;x) = 1 + \frac{\alpha+1}{\alpha}x + \frac{(\alpha+1)(\alpha+2)}{\alpha(\alpha+1)}\frac{x^2}{2!} + \cdots$$

$$= 1 + x + \frac{x}{\alpha} + \frac{x^2}{2!} + \frac{x^2}{\alpha} + \cdots$$

$$= 1 + \left(\frac{x}{\alpha} + \frac{x^2}{\alpha} + \cdots\right) + x + \frac{x^2}{2!} + \cdots$$

$$= \left(1 + x + \frac{x^2}{2!} + \cdots\right) + \frac{1}{\alpha}\left(x + x^2 + \frac{x^3}{2!} + \cdots\right)$$

$$= e^x + \frac{x}{\alpha}\left(1 + x + \frac{x^2}{2!} + \cdots\right)$$

$$= e^x + \frac{x}{\alpha}e^x$$

$$= \left(1 + \frac{x}{\alpha}\right)e^x.$$

Special Question 4:

$$\beta F(\alpha;\beta;x) = \beta F(\alpha-1;\beta;x) + xF(\alpha;\beta+1;x).$$

Solution.

We know that

$$F(\alpha;\beta;x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots \quad \text{---} (1)$$

Putting $\alpha = \alpha-1$ in (1), we get,

$$F(\alpha-1;\beta;x) = 1 + \frac{\alpha-1}{\beta}x + \frac{(\alpha-1)\alpha}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots$$

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$\Rightarrow \beta F(\alpha-1;\beta;x) = \beta + (\alpha-1)x + \frac{(\alpha-1)\alpha}{\beta+1}\frac{x^2}{2!} + \cdots$ ——— (2)

and

taking $\beta=\beta+1$ in (1), we get,

$F(\alpha;\beta+1;x) = 1 + \frac{\alpha}{\beta+1}x + \frac{\alpha(\alpha+1)}{(\beta+1)(\beta+2)}\frac{x^2}{2!} + \cdots$

$\Rightarrow xF(\alpha;\beta+1;x) = x + \frac{\alpha}{\beta+1}x^2 + \cdots$ ——— (3)

Adding (2) and (3), we get,

$\beta F(\alpha-1;\beta;x) + xF(\alpha;\beta+1;x) = \beta + (\alpha-1+1)x$

$+\left[\frac{(\alpha-1)\alpha}{(\beta+1)2!} + \frac{\alpha}{\beta+1}\right]x^2 + \cdots$

$= \beta + \alpha x + \frac{\alpha(\alpha-1)+2\alpha}{\beta+1}\frac{x^2}{2!} + \cdots$

$= \beta\left(1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \cdots\right)$

$= \beta F(\alpha;\beta;x)$.

Question 5:

$\alpha F(\alpha+1;\beta;x) - (\beta-1)F(\alpha;\beta-1;x) = (\alpha-\beta+1)F(\alpha;\beta;x)$

Self proceed.

Question 6:

$F(\alpha+1;\gamma;x) - F(\alpha;\gamma;x) = \frac{x}{\gamma}F(\alpha+1;\gamma+1;x)$.

Self proceed.

Question 7: (Pewal)

Prove that

$F(\alpha,\beta;\gamma;x) - F(\alpha;\beta;\gamma-1;x) = \frac{-\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1;\gamma+1;x)$

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Solution.

We know that

$$F(\alpha,\beta;\gamma;x) = 1+\frac{\alpha\beta}{\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+\cdots \quad \text{—— (1)}$$

$$F(\alpha;\beta;\gamma-1;x) = 1+\frac{\alpha\beta}{\gamma-1}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{(\gamma-1)\gamma}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{(\gamma-1)\gamma(\gamma+1)}\frac{x^3}{3!}+\cdots \quad \text{—— (2)}$$

Subtracting (2) from (1), we get,

$$F(\alpha,\beta;\gamma;x)-F(\alpha,\beta;\gamma-1;x) = \left(\frac{\alpha\beta}{\gamma}-\frac{\alpha\beta}{\gamma-1}\right)x+\left[\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}-\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma-1)}\right]\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)}\left(\frac{1}{\gamma+2}-\frac{1}{\gamma-1}\right)\frac{x^3}{3!}+\cdots$$

$$= \frac{\alpha\beta\gamma-\alpha\beta-\alpha\beta\gamma}{\gamma(\gamma-1)}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma}\left(\frac{\gamma-1-\gamma-1}{(\gamma+1)(\gamma-1)}\right)\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)}\left(\frac{\gamma-1-\gamma-2}{(\gamma+2)(\gamma-1)}\right)\frac{x^3}{3!}+\cdots$$

$$= \frac{-\alpha\beta x}{\gamma(\gamma-1)}\left(1+\frac{(\alpha+1)(\beta+1)}{\gamma+1}\frac{x}{1!}+\frac{(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}{(\gamma+1)(\gamma+2)}\frac{x^2}{2!}+\cdots\right)$$

$$= \frac{-\alpha\beta x}{\gamma(\gamma-1)}F(\alpha+1,\beta+1;\gamma+1;x)$$

Hence proved

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Question 8. 2016, 2018,

Prove that $${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}dt$$

Solution.

Since we know that

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\cdot\frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty}(\beta)_n\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\cdot\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}\cdot\frac{z^n}{n!} \quad \text{by Prop (1)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}(\beta)_n\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\cdot\frac{z^n}{n!}$$

$$\Rightarrow {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\sum_{n=0}^{\infty}(\beta)_n\frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(\gamma+n)}\frac{z^n}{n!} \longrightarrow (1)$$

By beta function $$\beta(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}$$

$$\Rightarrow \beta(\alpha+n,\gamma-\alpha) = \frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(\alpha+n+\gamma-\alpha)}$$

$$\Rightarrow \beta(\alpha+n,\gamma-\alpha) = \frac{\Gamma(\alpha+n)\Gamma(\gamma-\alpha)}{\Gamma(n+\gamma)} \longrightarrow (2)$$

Also

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt$$

$$\Rightarrow \beta(\alpha+n,\gamma-\alpha) = \int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}dt \longrightarrow (3)$$

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Comparing ② and ③, we get,

$$\frac{\Gamma(\alpha+n)\,\Gamma(\gamma-\alpha)}{\Gamma(n+\gamma)} = \int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1}\,dt \longrightarrow ④$$

Substituting ④ in ①, we get,

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)} \sum_{n=0}^{\infty} (\beta)_n \int_0^1 t^{\alpha+n-1}(1-t)^{\gamma-\alpha-1} \frac{z^n}{n!}\,dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)} \sum_{n=0}^{\infty} (\beta)_n \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \frac{t^n z^n}{n!}\,dt$$

$$\Rightarrow {}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1} \sum_{n=0}^{\infty} \frac{(\beta)_n (tz)^n}{n!}\,dt \longrightarrow ⑤$$

Since $\sum_{n=0}^{\infty} \frac{(\beta)_n (tz)^n}{n!} = (1-tz)^{-\beta}$

Therefore, ⑤ becomes

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}\,dt$$

as required

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CHAP 3:

STURM-LIOUVILLE (SL) System. (104)

2nd order linear homogeneous ordinary diff. equation

① Equation of simple harmonic motion

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

with conditions on $x(t)$ of the form

$$x(t) = x_0 \; ; \; \dot{x}(0) = 0$$

② Legendre's DE is

$$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$$

with conditions on $u(x)$ of the form

$u(\pm 1)$ are finite.

③ Bessel's DE is

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (k^2x^2 - \nu^2)u = 0$$

④ The Hermite DE is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0, \quad 0 \le x < \infty$$

⑤ The Laguerre DE is (لاگوایر)

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0, \; \dot{x} > 0$$

⑥ The Chebyshev DE is

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \alpha^2 y = 0 \; ; \; -1 < x < 1.$$

⑦ Airy's DE is

$$\frac{d^2y}{dx^2} + xy = 0 \; ; \; x \in \mathbb{R}.$$

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SL-Equation. 2014,

An equation of the form

$$\frac{d}{dx}\{p(x)\frac{dy}{dx}\} + q(x)y + \lambda s(x)y = 0 \quad \text{---} (1)$$

is called an SL equation if $p(x)$, $q(x)$, $p'(x)$ and $s(x)$ are real and continuous over an interval $[a,b]$.

If we define an operator L as

$$L \equiv \frac{d}{dx}\{p(x)\frac{d}{dx}\} + q(x).$$

Equation (1) can be written as

$$Ly + \lambda s(x)y = 0$$

where L is SL operator which is linear and self adjoint.

Regular-SL equation. 2014,

The SL equation $\frac{d}{dx}\{p(x)\frac{dy}{dx}\} + q(x)y + \lambda s(x)y = 0$ is called a regular SL equation over an interval $[a,b]$ if $p(x)$ and $s(x)$ do not vanish on $[a,b]$.

For example

(1) $u''(x) + \lambda u(x) = 0$

Here $p(x) = 1$; $q(x) = 0$ and $s(x) = 1$

This equation is regular over any interval

(2) $\frac{d}{dx}\{(1-x^2)\frac{du}{dx}\} + \lambda u = 0$

This is regular on any interval which does not contain 1 and -1.

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Singular SL equation.

The SL equation $\frac{d}{dx}\{p(x)\frac{dy}{dx}\}+q(x)y+\lambda s(x)y=0$ is called Singular over an interval $[a,b]$ if $p(x)$ or $s(x)$ vanishes at any point in the interval $[a,b]$.

SL-System.

An SL equation along with appropriate boundary conditions defines an SL-system.

Regular SL system. 2014,

A regular SL equation along with appropriate boundary conditions defines a regular SL system.

For example.

① $\frac{d^2u}{dx^2}+\lambda u=0$; $u(a)=0, u(b)=0$

② $\frac{d^2u}{dx^2}+\lambda u=0$; $u'(a)=0, u'(b)=0$

Periodic SL equation.

The DE $\frac{d}{dx}\{p(x)\frac{dy}{dx}\}+q(x)\frac{dy}{dx}+\lambda s(x)y=0$ is called periodic SL equation over an interval $[a,b]$ if $p(x)$, $q(x)$ and $s(x)$ are periodic functions of period $b-a$.

i.e. $p(a)=p(b)$; $q(a)=q(b)$; $s(a)=s(b)$.

Periodic SL system. 2014,

A periodic SL equation along with end

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Point conditions $u(a) = u(b)$ and $u'(a) = u'(b)$ defines a periodic SL-system.

For example.

(i) $u''(x) + \lambda u(x) = 0$

with end point conditions $u(\pi) = u(-\pi)$, $u'(\pi) = u'(-\pi)$ is a periodic SL-system with period 2π.

(ii) The Mathieu DE

$u'' + (\lambda + 16d\cos 2x)u = 0$

with end point conditions

$u(\pi) = u(-\pi)$ and $u'(-\pi) = u'(\pi)$

is a periodic SL system with period 2π.

Kinds of Boundary Conditions.

(1) Separated boundary Conditions.

The conditions over the interval $[a, b]$ of the form

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0$$
$$\beta_1 u(b) + \beta_2 u'(b) = 0$$

are called S.B.Cs.

(2) Dirichlet boundary conditions. (B.Cs of 1st form)

2010 The conditions over the interval $[a, b]$ of the form $u(a) = \alpha$; $u(b) = \beta$ are called D.B.Cs.

(3) Neumann Boundary Conditions (B.Cs of 2nd form)

The conditions over the interval $[a, b]$ of the form $u'(a) = \alpha$; $u'(b) = \beta$ are called N.B.Cs.

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④ Robin Boundary Conditions. (B.Cs of 3rd form)
The conditions over the interval $[a,b]$ of the form
$\alpha_1 u(a) + \alpha_2 u'(a) = C_1$
$\beta_1 u(b) + \beta_2 u'(b) = C_2$
are called R.B.Cs.

⑤ Mixed Boundary conditions.
The conditions over the interval $[a,b]$ of the form
$\alpha_1 u(a) + \alpha_2 u'(b) = C_1$
$\beta_1 u(a) + \beta_2 u'(b) = C_2$
are called M.B.Cs.

Eigensolutions and Eigenvalues of an SL-System. 2014, S/2018

A non zero solution $u(x)$ of an SL System (regular or periodic) is said to be an eigensolution or eigenfunction corresponding a value of the parameter λ involved in the SL equation.

The value of λ for which we have an eigensolution is called an eigenvalue.

For example.

(i) $y'' + \lambda y = 0$; $y(0) = 0$; $y(l) = 0$
has eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and eigenfunction
$y_n(x) = a_n \sin\frac{n\pi}{l}x$
where $n = 1, 2, 3, \cdots$

(ii) $y'' + \lambda y = 0$; $y'(0) = y'(l) = 0$

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has eigenvalues $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and

eigenfunctions $y_n = a_n \cos\left(\frac{n\pi}{l}x\right)$

where $n = 0, 1, 2, 3, \ldots$

Remark:

Eigenvalue can be zero but eigenfunction or eigensolution can never be zero.

Example 1: 2012,

Find the eigenvalues and eigenfunctions of the regular SL system

$u'' + \lambda u = 0$; $u(0) = 0$, $u(\pi) = 0$

Solution.

Given that

$u'' + \lambda u = 0$; $u(0) = 0$, $u(\pi) = 0$ ——— (1)

Therefore, the characteristic equation of (1) is, $D^2 + \lambda = 0$

$\Rightarrow D = \pm i\sqrt{\lambda}$

So the general solution is

$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ ——→ (2)

$u(0) = 0 \Rightarrow 0 = A\cos 0^\circ + B\sin 0^\circ$

$\Rightarrow A = 0$

$u(\pi) = 0 \Rightarrow 0 = 0 + B\sin\sqrt{\lambda}\pi$

$\Rightarrow B\sin\sqrt{\lambda}\pi = 0$

The choice $B = 0$ gives a trivial solution, so this choice is not acceptable as we want to find non-trivial solution so

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$\sin\sqrt{\lambda}\pi = 0$

$\Rightarrow \sqrt{\lambda}\pi = \pm n\pi \quad ; n = \pm 1, \pm 2, \cdots$

$\Rightarrow \sqrt{\lambda} = n$

$\Rightarrow \lambda = \lambda_n = n^2 \quad ; n = \pm 1, \pm 2, \cdots$

② $\Rightarrow u_n(x) = B_n \sin\sqrt{\lambda_n}\, x$

where $\lambda_n = n^2$, $n = \pm 1, \pm 2, \cdots$

Hence eigenfunctions are

$u_n = B_n \sin\sqrt{\lambda_n}\, x$ and

eigenvalues are $\lambda_n = n^2$, $n = \pm 1, \pm 2, \pm 3, \cdots$

Example 2: 2017, (Special)

Determine eigenvalues of the system $u'' + \lambda u = 0$ with $u(0) = u(\pi)$, $u'(0) = 2u'(\pi)$.

Solution.

Given that $u'' + \lambda u = 0$ ——— ①

with $u(0) = u(\pi)$, $u'(0) = 2u'(\pi)$ ——→ ②

So the general solution is

$u(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$ ——— ③

$\Rightarrow u(0) = A\cos 0^\circ + B\sin 0^\circ$

$\Rightarrow u(0) = A$

and $u(\pi) = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

using 1st end point conditions of ②, we have

$A = A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi$

$\Rightarrow A(\cos\sqrt{\lambda}\pi - 1) + B\sin\sqrt{\lambda}\pi = 0$ ——— ④

Diff. ③, w.r.t 'x', we get,

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$$u'(x) = -A\sin\sqrt{\lambda}x \cdot \sqrt{\lambda} + B\cos\sqrt{\lambda}x \cdot \sqrt{\lambda}$$

$$\Rightarrow u'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$x=0 \Rightarrow u'(0) = -A\sqrt{\lambda}\sin 0^\circ + B\sqrt{\lambda}\cos 0^\circ$$

$$\Rightarrow u'(0) = B\sqrt{\lambda}$$

and $x=\pi \Rightarrow u'(\pi) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

using second end point condition of ②, we get,

$$u'(0) = 2u'(\pi)$$

$$\Rightarrow B\sqrt{\lambda} = -2A\sqrt{\lambda}\sin\sqrt{\lambda}\pi + 2B\sqrt{\lambda}\cos\sqrt{\lambda}\pi$$

$$\Rightarrow B = -2A\sin\sqrt{\lambda}\pi + 2B\cos\sqrt{\lambda}\pi$$

$$\Rightarrow 2A\sin\sqrt{\lambda}\pi + B(1-2\cos\sqrt{\lambda}\pi) = 0 \longrightarrow ⑤$$

From ④ and ⑤, we can write,

$$\begin{bmatrix} \cos\sqrt{\lambda}\pi - 1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1-2\cos\sqrt{\lambda}\pi \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $A=0, B=0$ give trivial solution.
For non trivial solutions, we must have

$$\begin{vmatrix} \cos\sqrt{\lambda}\pi - 1 & \sin\sqrt{\lambda}\pi \\ 2\sin\sqrt{\lambda}\pi & 1-2\cos\sqrt{\lambda}\pi \end{vmatrix} = 0$$

$$\Rightarrow (\cos\sqrt{\lambda}\pi - 1)(1-2\cos\sqrt{\lambda}\pi) - \sin\sqrt{\lambda}\pi \cdot 2\sin\sqrt{\lambda}\pi = 0$$

$$\Rightarrow \cos\sqrt{\lambda}\pi - 2\cos^2\sqrt{\lambda}\pi - 1 + 2\cos\sqrt{\lambda}\pi - 2\sin^2\sqrt{\lambda}\pi = 0$$

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$\Rightarrow 3\cos\sqrt{\lambda}\pi - 1 - 2(\cos^2\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi) = 0$

$\Rightarrow 3\cos\sqrt{\lambda}\pi - 3 = 0$

$\Rightarrow \cos\sqrt{\lambda}\pi = 1$

$\Rightarrow \sqrt{\lambda}\pi = 2n\pi \quad ; n = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \sqrt{\lambda} = 2n$

$\Rightarrow \lambda = \lambda_n = 4n^2 \quad ; \quad n = 0, \pm 1, \pm 2, \cdots$

are eigen values and eigenfunctions are

$u_n = A_n \cos 2nx + B_n \sin 2nx$.

Example 3: 2016,

Show that if $u(x)$ and $v(x)$ are periodic solutions of the Mathieu equation with period π, having distinct eigenvalues, then

$$\int_a^b u(x)\, v(x)\, dx = 0$$

Solution.

The Mathieu equation is

$$y'' + \lambda y + 16d(\cos 2x)\, y = 0 \quad \text{——} \quad (1)$$

with end point conditions

$$y(0) = y(\pi), \quad y'(0) = y'(\pi) \quad \text{——} \quad (2)$$

As given that u and v are solutions of (1) corresponding to distinct eigenvalues, say, $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively.

So $u'' + \lambda_1 u + 16d(\cos 2x)\, u = 0$ —— (3)

with $u(0) = u(\pi), \quad u'(0) = u'(\pi)$ —— (4)

and

$$v'' + \lambda_2 v + 16d(\cos 2x)\, v = 0 \quad \text{——} \quad (5)$$

with $v(0) = v(\pi), \quad v'(0) = v'(\pi)$ —— (6)

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Xing (3) by v and (5) by u and then subtracting, we get,

$$u''v + \lambda_1 uv + 16d(\cos 2x)uv - uv'' - \lambda_2 uv - 16d(\cos 2x)uv = 0$$

$$\Rightarrow u''v - uv'' + (\lambda_1 - \lambda_2)uv = 0$$

$$\Rightarrow (\lambda_1 - \lambda_2)uv = uv'' - u''v$$

Integrating w.r.t. x from 0 to π, we get,

$$\int_0^\pi (\lambda_1 - \lambda_2)uv\,dx = \int_0^\pi (uv'' - u''v)\,dx$$

$$\Rightarrow (\lambda_1 - \lambda_2)\int_0^\pi uv\,dx = \int_0^\pi \frac{d}{dx}(uv' - u'v)\,dx$$

$$= [uv' - u'v]_0^\pi$$

$$= u(\pi)v'(\pi) - u'(\pi)v(\pi) - u(0)v'(0) + u'(0)v(0)$$

using (4) and (6), in above relation, we get,

$$(\lambda_1 - \lambda_2)\int_0^\pi uv\,dx = u(0)v'(0) - u'(0)v(0) - u(0)v'(0) + u'(0)v(0)$$

$$\Rightarrow (\lambda - \lambda_2)\int_0^\pi uv\,dx = 0$$

Since $\lambda_1 \neq \lambda_2$, therefore, $\lambda_1 - \lambda_2 \neq 0$

$$\Rightarrow \int_0^\pi uv\,dx = 0$$

as required

Theorem 1 : 2010,

If $u_m(x)$ and $u_n(x)$ are solutions of a periodic SL system corresponding to distinct eigenvalues λ_m and λ_n of the parameter λ, then $u_m(x)$ and $u_n(x)$ are orthogonal w.r.t. the weight function $s(x)$.

i.e. $\int_a^b s(x)\, u_m(x)\, u_n(x)\, dx = 0$, $m \neq n$.

Proof.

Consider a periodic SL system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda s(x)u = 0 \; ; x \in [a,b] \longrightarrow (1)$$

with $u(a) = u(b)$, $u'(a) = u'(b)$ $\longrightarrow$ (2)

having eigensolutions $u_m(x)$ and $u_n(x)$ corresponding to distinct eigenvalues λ_m and λ_n respectively.

As $u_m(x)$ and $u_n(x)$ are eigensolutions of periodic SL-system so (1) and (2) imply

$$\frac{d}{dx}\left\{p(x)\frac{du_m}{dx}\right\} + q(x)u_m + \lambda_m s(x)u_m(x) = 0 \longrightarrow (3)$$

with $u_m(a) = u_m(b)$, $u_m'(a) = u_m'(b)$ $\longrightarrow$ (4)

and

$$\frac{d}{dx}\left\{p(x)\frac{du_n}{dx}\right\} + q(x)u_n + \lambda_n s(x)u_n(x) = 0 \longrightarrow (5)$$

with $u_n(a) = u_n(b)$, $u_n'(a) = u_n'(b)$ $\longrightarrow$ (6)

Xing (3) by u_n and (5) by u_m and then subtracting, we get,

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$$u_n \frac{d}{dx}\{p(x)u_m'(x)\} + q(x)u_m u_n + \lambda_m \rho(x) u_m u_n$$

$$- u_m \frac{d}{dx}\{p(x)u_n'(x)\} - q(x)u_n u_m - \lambda_n \rho(x) u_n u_m = 0$$

$$\Rightarrow u_n p'(x) u_m'(x) + u_n p(x) u_m''(x) + \lambda_m \rho(x) u_m u_n$$

$$- u_m p'(x) u_n'(x) - u_m p(x) u_n''(x) - \lambda_n \rho(x) u_m u_n = 0$$

$$\Rightarrow p'(u_n u_m' - u_m u_n') + (u_n u_m'' - u_m u_n'')p$$

$$+ (\lambda_m - \lambda_n)\rho(x) u_m u_n = 0$$

$$\Rightarrow p'(u_n u_m' - u_m u_n') + p'(u_n u_m'' - u_m u_n'')$$

$$+ (\lambda_m - \lambda_n)\rho(x) u_m u_n = 0$$

$$\Rightarrow (\lambda_m - \lambda_n)\rho(x) u_m u_n = p'(u_m u_n' - u_n u_m')$$

$$+ p(u_m u_n'' - u_n u_m'')$$

$$\Rightarrow (\lambda_m - \lambda_n)\rho(x) u_m u_n = \frac{d}{dx}[p(u_m u_n' - u_m' u_n)]$$

Integrating w.r.t. x, from a to b, we get

$$(\lambda_m - \lambda_n)\int_a^b \rho(x) u_m u_n \, dx = \int_a^b \frac{d}{dx}[(u_m u_n' - u_m' u_n)p]\, dx$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b \rho(x) u_m u_n \, dx = [p(u_m u_n' - u_m' u_n)]_a^b$$

$$= p(b)[u_m(b)u_n'(b) - u_m'(b)u_n(b)]$$

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$-p(a)[u_m(a)u_n'(a) - u_m'(a)u_n(a)]$

using (4), (6) and $p(a) = p(b)$, as system is periodic, we get,

$$(\lambda_m - \lambda_n)\int_a^b r(x)u_m u_n dx = p(b)[u_m(b)u_n'(b) - u_m'(b)u_n(b)]$$

$$- p(b)[u_m(b)u_n'(b) - u_m'(b)u_n(b)]$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b r(x)u_m u_n dx = 0$$

Since λ_m and λ_n are distinct, therefore,

$\lambda_m - \lambda_n \neq 0$.

$$\Rightarrow \int_a^b r(x)u_m(x)u_n(x)dx = 0$$ as required.

Theorem 2: 2012,

If $u_m(x)$ and $u_n(x)$ are solutions of a regular SL system corresponding to distinct eigenvalues λ_m and λ_n of the parameter λ, then $u_m(x)$ and $u_n(x)$ are orthogonal w.r.t. the weight function $r(x)$.

i.e. $\int_a^b u_m(x)u_n(x)r(x)dx = 0$, $m \neq n$

Proof.

Consider the regular SL system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda r(x)u = 0, \quad x \in [a,b] \longrightarrow (1)$$

with $\alpha u(a) + \alpha' u'(a) = 0$ ——— (2)

$\beta u(b) + \beta' u'(b) = 0$ ——— (3)

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having eigensolutions u_m and u_n corresponding to distinct eigenvalues λ_m and λ_n respectively.

As $u_m(x)$ and $u_n(x)$ are eigensolutions of regular SL-system so ①, ②, and ③ imply

$$\frac{d}{dx}\{p(x)u_m'(x)\} + q(x)u_m(x) + \lambda_m r(x)u_m(x) = 0 \longrightarrow ④$$

with
$$\left.\begin{aligned} \alpha u_m(a) + \alpha' u_m'(a) &= 0 \\ \beta u_m(b) + \beta' u_m'(b) &= 0 \end{aligned}\right\} \longrightarrow ⑤$$

and $$\frac{d}{dx}\{p(x)u_n'(x)\} + q(x)u_n(x) + \lambda_n r(x)u_n(x) = 0 \longrightarrow ⑥$$

with
$$\left.\begin{aligned} \alpha u_n(a) + \alpha' u_n'(a) &= 0 \\ \beta u_n(b) + \beta' u_n'(b) &= 0 \end{aligned}\right\} \longrightarrow ⑦$$

$\times$ing ④ by u_n and ⑥ by u_m, and then subtracting, we get,

$$u_n\frac{d}{dx}\{p(x)u_m'(x)\} + q(x)u_m u_n + \lambda_m r(x)u_m u_n$$
$$- u_m\frac{d}{dx}\{p(x)u_n'(x)\} - q(x)u_m u_n - \lambda_n r(x)u_m u_n = 0$$

$$\Rightarrow u_n p'(x)u_m'(x) + u_n p(x)u_m''(x) + \lambda_m r(x)u_m u_n$$
$$- u_m p'(x)u_n'(x) - u_m p(x)u_n''(x) - \lambda_n r(x)u_m u_n = 0$$

$$\Rightarrow p'(u_n u_m' - u_m u_n') + p(u_n u_m'' - u_m u_n'')$$
$$+ (\lambda_m - \lambda_n) r(x)u_m u_n = 0$$

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$$\Rightarrow (\lambda_m - \lambda_n) r(x) u_m u_n = p'(u_m u_n' - u_n u_m') + p(u_m u_n'' - u_n u_m'')$$

$$\Rightarrow (\lambda_m - \lambda_n) r(x) u_m u_n = \frac{d}{dx}[p(u_m u_n' - u_m' u_n)]$$

Integrating w.r.t. x from a to b, we get,

$$(\lambda_m - \lambda_n)\int_a^b r(x) u_m u_n dx = \int_a^b \frac{d}{dx}[p(u_m u_n' - u_m' u_n)]dx$$

$$= [p(u_m u_n' - u_m' u_n)]_a^b$$

$$= p(b)[u_m(b) u_n'(b) - u_m'(b) u_n(b)] - p(a)[u_m(a) u_n'(a) - u_m'(a) u_n(a)]$$

using (5) and (7) in above relation, we get

$$(\lambda_m - \lambda_n)\int_a^b r(x) u_m u_n dx = p(b)\left[-\tfrac{\beta}{\beta'}, u_m(b) u_n(b) + \tfrac{\beta}{\beta'}, u_m(b) u_n(b)\right] - p(a)\left[-\tfrac{\alpha}{\alpha'}, u_m(a) u_n(a) + \tfrac{\alpha}{\alpha'} u_m(a) u_n(a)\right]$$

$$\Rightarrow (\lambda_m - \lambda_n)\int_a^b r(x) u_m u_n dx = 0$$

Since λ_m and λ_n are distinct, therefore,

$$\lambda_m - \lambda_n \neq 0$$

$$\Rightarrow \int_a^b r(x) u_m(x) u_n(x) dx = 0$$

as required result

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Theorem 3: 2011, 2013, 2015,
The eigenvalues of regular SL-system are real.

Proof.
Let $\lambda = \lambda_1 + i\lambda_2$ be an eigenvalue of a regular SL-system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda s(x)u(x) = 0 \longrightarrow (1)$$

with end point conditions

$$\alpha u(a) + \alpha_1 u'(a) = 0 \longrightarrow (2)$$
$$\beta u(b) + \beta_1 u'(b) = 0$$

We have to show that $\lambda_2 = 0$.
As we have suppose a complex eigenvalue so the corresponding eigenfunction will be complex.
Therefore, we may write $u = \varphi + i\psi$
Thus,

$$(1) \Rightarrow \frac{d}{dx}\left\{p(x)\frac{d(\varphi+i\psi)}{dx}\right\} + q(x)(\varphi+i\psi) + (\lambda_1+i\lambda_2)s(x)(\varphi+i\psi) = 0 \longrightarrow (3)$$

with $\alpha[\varphi(a) + i\psi(a)] + \alpha_1[\varphi'(a) + i\psi'(a)] = 0$
$\beta[\varphi(b) + i\psi(b)] + \beta_1[\varphi'(b) + i\psi'(b)] = 0 \longrightarrow (4)$

Equating real and imaginary parts on both sides, we get,

$$(P\varphi')' + q\varphi + \lambda_1 s\varphi - \lambda_2 s\psi = 0 \longrightarrow (5)$$

$$(P\psi')' + q\psi + \lambda_1 s\psi + \lambda_2 s\varphi = 0 \longrightarrow (6)$$

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with $\alpha\varphi(a) + \alpha_1\varphi'(a) = 0$ ⟶ (7)

$\beta\varphi(b) + \beta_1\varphi'(b) = 0$

and

$\alpha\psi(a) + \alpha_1\psi'(a) = 0$ ⟶ (8)

$\beta\psi(b) + \beta_1\psi'(b) = 0$

Xing (5) by ψ and (6) by φ, and then Subtracting, we have,

$\psi(p\varphi')' + q\varphi\psi + \lambda_1 s\varphi\psi - \lambda_2 s\psi^2$

$-\varphi(p\psi')' - q\varphi\psi - \lambda_1 s\varphi\psi - \lambda_2 s\varphi^2 = 0$

$\Rightarrow \psi(p\varphi')' - \varphi(p\psi')' - \lambda_2 s(\varphi^2+\psi^2) = 0$

$\Rightarrow \lambda_2 s(\varphi^2+\psi^2) = \psi(p\varphi')' - \varphi(p\psi')'$

$= \psi(p\varphi'' + p'\varphi') - \varphi(p\psi'' + p'\psi')$

$= p(\psi\varphi'' - \varphi\psi'') + p'(\psi\varphi' - \varphi\psi')$

$\Rightarrow \lambda_2 s(\varphi^2+\psi^2) = \frac{d}{dx}[p(\psi\varphi' - \varphi\psi')]$

Integrating w.r.t. 'x' from 'a' to 'b', we get,

$$\int_a^b \lambda_2 s(x)(\varphi^2+\psi^2)\,dx = \int_a^b \frac{d}{dx}[p(\psi\varphi' - \varphi\psi')]\,dx$$

$$\Rightarrow \lambda_2 \int_a^b s(\varphi^2+\psi^2)\,dx = [p(\psi\varphi' - \varphi\psi')]_a^b$$

$= p(b)[\psi(b)\varphi'(b) - \varphi(b)\psi'(b)]$

$- p(a)[\psi(a)\varphi'(a) - \varphi(a)\psi'(a)].$ ⟶ (9)

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using (7) and (8) in (9), we get,

$$\lambda_2 \int_a^b r(\varphi^2+\psi^2)\,dx = p(b)\left[-\frac{\beta}{\beta_1}\varphi(b)\psi(b)+\frac{\beta}{\beta_1}\varphi(b)\psi(b)\right] - p(a)\left[-\frac{\alpha}{\alpha_1}\varphi(a)\psi(a)+\frac{\alpha}{\alpha_1}\varphi(a)\psi(a)\right]$$

$$\Rightarrow \lambda_2 \int_a^b r(\varphi^2+\psi^2)\,dx = 0$$

As $\varphi+i\psi$ is an eigensolution so φ and ψ can never be zero simultaneously.

$$\Rightarrow \varphi^2+\psi^2 \neq 0$$

Also for a regular SL-system $r(x)$ does not vanish (i.e. $r(x)\neq 0$) on $[a,b]$.

$\Rightarrow$ $r(x)$ does not change its sign on $[a,b]$.

Therefore, $\int_a^b r(\varphi^2+\psi^2)\,dx \neq 0$.

$$\Rightarrow \lambda_2 = 0$$

Hence $\lambda = \lambda_1 + i\lambda_2 = \lambda_1 + i0$

which is real.

Theorem 4.

The eigenvalues of periodic SL-system are real.

Proof.

Let $\lambda = \lambda_1 + i\lambda_2$ be an eigenvalue of a periodic SL system

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u(x) + \lambda r(x)u(x) = 0 \quad \text{---} (1)$$

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with $u(a) = u(b)$; $u'(a) = u'(b)$ —— (2)

We have to show that $\lambda_2 = 0$.

As we have suppose a complex eigenvalue so the corresponding eigenfunction will be complex.

Therefore, we may write $u = \varphi + i\psi$

Thus

$$(1) \Rightarrow \frac{d}{dx}\{p(x)\frac{d}{dx}(\varphi+i\psi)\} + q(x)(\varphi+i\psi) + (\lambda_1+i\lambda_2)s(\varphi+i\psi) = 0 \longrightarrow (3)$$

with $\varphi(a) + i\psi(a) = \varphi(b) + i\psi(b)$

$\varphi'(a) + i\psi'(a) = \varphi'(b) + i\psi'(b)$. ——— (4)

Equating real and imaginary parts on both sides, we get,

$$(p\varphi')' + q\varphi + \lambda_1 s\varphi - \lambda_2 s\psi = 0 \longrightarrow (5)$$

$$(p\psi')' + q\psi + \lambda_1 s\psi + \lambda_2 s\varphi = 0 \longrightarrow (6)$$

with

$$\varphi(a) = \varphi(b) \; ; \; \psi(a) = \psi(b) \longrightarrow (7)$$

and $\varphi'(a) = \varphi'(b)$; $\psi'(a) = \psi'(b)$ ——— (8)

Xing (5) by ψ and (6) by φ, and then subtracting, we get,

$$\psi(p\varphi')' + q\varphi\psi + \lambda_1 s\varphi\psi - \lambda_2 s\psi^2$$
$$- \varphi(p\psi')' - q\varphi\psi - \lambda_1 s\varphi\psi - \lambda_2 s\varphi^2 = 0$$

$$\Rightarrow \psi(p\varphi')' - \varphi(p\psi')' - \lambda_2 s(\varphi^2+\psi^2) = 0$$

$$\Rightarrow \lambda_2 s(\varphi^2+\psi^2) = \psi(p\varphi'' + p'\varphi') - \varphi(p\psi'' + p'\psi')$$

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$$= p(\psi\varphi'' - \varphi\psi'') + p'(\psi\varphi' - \varphi\psi')$$

$$\Rightarrow \lambda_2 r(\varphi^2+\psi^2) = \frac{d}{dx}[p(\psi\varphi' - \varphi\psi')]$$

Integrating w.r.t. x, from a to b, we have,

$$\int_a^b \lambda_2 r(\varphi^2+\psi^2)\,dx = \int_a^b \frac{d}{dx}[p(\psi\varphi' - \varphi\psi')]\,dx$$

$$\Rightarrow \lambda_2 \int_a^b r(\varphi^2+\psi^2)\,dx = [p(\psi\varphi' - \varphi\psi')]_a^b$$

$$= p(b)[\psi(b)\varphi'(b) - \varphi(b)\psi'(b)] - p(a)[\psi(a)\varphi'(a) - \varphi(a)\psi'(a)] \rightarrow (9)$$

Using (7), (8) and $p(a) = p(b)$, as system is periodic, in (9), we get,

$$\lambda_2 \int_a^b r(\varphi^2+\psi^2)\,dx = p(b)[\psi(b)\varphi'(b) - \varphi(b)\psi'(b)] - p(b)[\psi(b)\varphi'(b) - \varphi(b)\psi'(b)]$$

$$\Rightarrow \lambda_2 \int_a^b r(\varphi^2+\psi^2)\,dx = 0$$

As $\varphi + i\psi$ is an eigensolution so φ and ψ can never be zero simultaneously.

$$\Rightarrow \varphi^2 + \psi^2 \neq 0$$

Also for a periodic SL system $r(x)$ is a periodic function of period '$b-a$' on $[a, b]$.

$$\Rightarrow r(x) \neq 0.$$

Therefore, $\int_a^b r(\varphi^2+\psi^2)\,dx \neq 0$

$$\Rightarrow \lambda_2 = 0$$

Hence $\lambda = \lambda_1 + i\lambda_2 = \lambda_1 + i0$ which is real.

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Exercise

Question 1:

Find all eigenvalues and eigenfunctions for $y'' + \lambda y = 0$ with $y(0) = 0,\ 3y(\pi) - y'(\pi) = 0$.

Solution.

The characteristic equation is

$$D^2 + \lambda = 0$$

$$\Rightarrow D^2 = -\lambda \qquad \text{——— (1)}$$

Case I. $\lambda = 0$.

(1) $\Rightarrow D^2 = 0 \ ; \ D = 0, 0.$ (real repeated)

The general solution is

$$y = (C_1 + C_2 x)e^{0x} = C_1 + C_2 x$$

$y(0) = 0 \Rightarrow 0 = C_1 \Rightarrow \boxed{C_1 = 0}$

So $y = C_2 x$

$y' = C_2$

$3y(\pi) - y'(\pi) = 0$

$\Rightarrow 3C_2\pi - C_2 = 0 \Rightarrow C_2(3\pi - 1) = 0$

Since $3\pi - 1$ can never be zero.

Therefore $\boxed{C_2 = 0}$

Thus $y = 0$ (a trivial solution).

Hence in this case we have no eigenvalue and no eigensolution.

Case II: $\lambda < 0$.

we may take $\lambda = -\mu^2$ where $\mu \in \mathbb{R} \setminus \{0\}$.

(1) $\Rightarrow D^2 = -(-\mu^2) = \mu^2$

$\Rightarrow D = \pm\mu$ (real, ~~repeated~~ distinct)

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The general solution is

$$y = C_1 e^{\mu x} + C_2 e^{-\mu x}.$$

$y(0) = 0$

$\Rightarrow 0 = C_1 + C_2 \Rightarrow C_1 = -C_2 \longrightarrow (2)$

$$y' = C_1 \mu e^{\mu x} - C_2 \mu e^{-\mu x}.$$

$3y(\pi) - y'(\pi) = 0$

$$\Rightarrow 3C_1 e^{\mu\pi} + 3C_2 e^{-\mu\pi} - C_1 \mu e^{\mu\pi} + C_2 \mu e^{-\mu\pi} = 0$$

using (2), we have,

$$-3C_2 e^{\mu\pi} + 3C_2 e^{-\mu\pi} + C_2 \mu e^{-\mu\pi} + C_2 \mu e^{-\mu\pi} = 0$$

$$\Rightarrow C_2[-3(e^{\mu\pi} - e^{-\mu\pi}) + \mu(e^{\mu\pi} + e^{-\mu\pi})] = 0$$

$$\Rightarrow C_2[-6\sinh\mu\pi + 2\mu\cosh\mu\pi] = 0$$

Since $-6\sinh\mu\pi + 2\mu\cosh\mu\pi \neq 0$.

Therefore, $\boxed{C_2 = 0}$

(2) $\Rightarrow \boxed{C_1 = 0}$

Thus $y = 0$, (a trivial solution).

Hence in this case we have no eigenvalue and no eigen solution.

Case III. $\lambda > 0$.

we may take $\lambda = \mu^2$, where $\mu \in \mathbb{R} \setminus \{0\}$.

(1) $\Rightarrow D^2 = -\mu^2$

$\Rightarrow D = \pm i\mu$ (Complex)

The general solution is

$$y = C_1 \sin\mu x + C_2 \cos\mu x$$

$y(0) = 0$

$\Rightarrow 0 = 0 + C_2 \Rightarrow \boxed{C_2 = 0}$

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So $y(x) = G \sin \mu x$

$y'(x) = G\mu \cos \mu x$

$3y(\pi) - y'(\pi) = 0$

$\Rightarrow 3G \sin \mu\pi - G\mu \cos \mu\pi = 0$

$\Rightarrow G(3 \sin \mu\pi - \mu \cos \mu\pi) = 0$

For non trivial solution, we have $G \neq 0$.

$\Rightarrow 3 \sin \mu\pi - \mu \cos \mu\pi = 0$

$\Rightarrow 3 \sin \mu\pi = \mu \cos \mu\pi$

$\Rightarrow 3 \tan \mu\pi = \mu$

$\Rightarrow \tan \mu\pi = \mu/3 \longrightarrow (3)$

Thus eigensolutions are $y_n = G_n \sin \mu_n x$ with eigenvalues $\lambda_n = \mu_n^2$; $n = 0, \pm 1, \pm 2, \cdots$

μ_n satisfies the equation

$$\tan \mu_n \pi = \frac{\mu_n}{3} \quad ; n = 0, \pm 1, \pm 2, \cdots$$

Special Question 2: 2016,

Determine all the real eigenvalues and eigenfunctions for

$$(xy')' + \frac{\lambda}{x} y = 0, \quad 1 < x < e.$$

with end point conditions $y(1) = 0$, $y(e) = 0$.

Solution.

Given that $(xy')' + \frac{\lambda}{x} y = 0 \longrightarrow (1)$

Therefore,

$$xy'' + y' + \frac{\lambda}{x} y = 0$$

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$\Rightarrow x^2y'' + xy' + \lambda y = 0 \longrightarrow (2)$

which is a cauchy-Euler equation.

To solve it we make the subsitution;

$x = e^t \Rightarrow t = \ln x$

$xD = \Delta$ and $x^2D^2 = \Delta(\Delta - 1)$

$(2) \Rightarrow x^2D^2y + xDy + \lambda y = 0 \quad \because D = \frac{d}{dx}$

$\Rightarrow \Delta(\Delta-1)y + \Delta y + \lambda y = 0$

$\Rightarrow (\Delta(\Delta-1) + \Delta + \lambda)y = 0$

$\Rightarrow (\Delta^2 - \Delta + \Delta + \lambda)y = 0$

$\Rightarrow (\Delta^2 + \lambda)y = 0 \longrightarrow (3)$

The characteristic equation is

$\Delta^2 + \lambda = 0$

$\Rightarrow \Delta^2 = \pm\lambda \longrightarrow (4)$

Case I: $\lambda = 0$

$(4) \Rightarrow \Delta^2 = 0, 0$ (real, repeated)

The general solution is $y(t) = (C_1 + C_2 t)e^{0t}$.

$y = C_1 + C_2 \ln x \quad \because C_1 + C_2 t = y$

Applying end point conditions, we get,

$y(1) = 0 \Rightarrow 0 = C_1 + C_2 \ln(0)$

$\Rightarrow \boxed{C_1 = 0}$ so $y = C_2 \ln x$

$y(e) = 0 \Rightarrow 0 = C_2 \ln e$

$\Rightarrow \boxed{C_2 = 0}$

Thus $y = 0$ (a trivial solution).

Hence in this case we have no eigenvalue and eigensolution.

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Case II: $\lambda < 0$

We may take $\lambda = -\mu^2$ where $\mu \in \mathbb{R} \setminus \{0\}$.

④ $\Rightarrow \Delta^2 = -(-\mu^2) = \mu^2$

$\Delta = \pm \mu$ (real, distinct)

The general solution is

$$y = c_1 e^{\mu t} + c_2 e^{-\mu t}$$
$$= c_1 e^{\mu \ln x} + c_2 e^{-\mu \ln x}$$
$$= c_1 e^{\mu \ln x} + c_2 e^{\ln x^{-\mu}} = c_1 e^{\ln x^{\mu}} + c_2 e^{\ln x^{-\mu}}$$

$$\Rightarrow y = c_1 x^{\mu} + c_2 x^{-\mu}$$

Applying end point conditions, we get,

$y(1) = 0 \Rightarrow 0 = c_1 + c_2$

$\Rightarrow c_1 = -c_2 \longrightarrow$ (5)

$y = -c_2 x^{\mu} + c_2 x^{-\mu} = c_2 (x^{-\mu} - x^{\mu})$

$y(e) = 0$

$\Rightarrow 0 = c_2 (e^{-\mu} - e^{\mu})$

For non trivial solution

$e^{-\mu} - e^{\mu} = 0$

$\Rightarrow e^{-\mu} = e^{\mu}$

$\Rightarrow -\mu = \mu$

$\Rightarrow 2\mu = 0 \Rightarrow \mu = 0$

but $\mu > 0$ $\because \mu \in \mathbb{R} \setminus \{0\}$.

Which is i.e $\mu = 0$ is not possible.

So $\boxed{c_2 = 0}$

(5) $\Rightarrow \boxed{c_1 = 0}$

Thus $y = 0$ a trivial solution.

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Case III: $\lambda > 0$.

We may take $\lambda = \mu^2$ where $\mu \in \mathbb{R} \setminus \{0\}$.

④ $\Rightarrow \Delta^2 = -\mu^2$

$\Rightarrow \Delta = \pm i\mu$ (complex).

The general solution is

$y = C_1 \sin \mu t + C_2 \cos \mu t = C_1 \sin \mu \ln x + C_2 \cos \mu \ln x$

$y(1) = 0$

$\Rightarrow 0 = C_1 \sin \mu \ln 1 + C_2 \cos \mu \ln 1$

$\Rightarrow 0 = C_1 (0) + C_2$

$\Rightarrow \boxed{C_2 = 0}$

So $y = C_1 \sin(\mu \ln x)$

$y(e) = 0 \Rightarrow 0 = C_1 \sin(\mu \ln e)$

$\Rightarrow 0 = C_1 \sin \mu$

For non trivial solution, we have,

$C_1 \neq 0$

$\Rightarrow \sin \mu = 0 \Rightarrow \mu = n\pi \; ; n = 1, 2, 3, \cdots$.

Thus eigenvalues are $\lambda_n = \mu_n^2 = n^2\pi^2$

and eigen solutions are

$y_n = C_n \sin(\mu_n \ln x)$.

Question 3:

Find eigenvalues and eigenfunctions for $u'' + \lambda u = 0$

with $u(0) = u(\pi) + u'(\pi) = 0$

solution.

Given that $u'' + \lambda u = 0$ ——— (1)

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The characteristic equation is

$$D^2+\lambda=0$$

$$\Rightarrow D^2=-\lambda \longrightarrow (2)$$

$$\Rightarrow D=\pm i\sqrt{\lambda} \quad \text{(complex)}$$

The general solution is

$$u=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x$$

Applying boundary conditions,

$u(0)=0 \Rightarrow \boxed{0=C_1}$

$$u=C_2\sin\sqrt{\lambda}x$$

$$u'=C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$u(\pi)+u'(\pi)=0 \Rightarrow 0=C_2\sin\sqrt{\lambda}\pi+C_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi$

$$\Rightarrow C_2(\sin\sqrt{\lambda}\pi+\sqrt{\lambda}\cos\sqrt{\lambda}\pi)=0$$

For non trivial solution $C_2\neq 0$.

$$\Rightarrow \sin\sqrt{\lambda}\pi+\sqrt{\lambda}\cos\sqrt{\lambda}\pi=0$$

$$\Rightarrow \sin\sqrt{\lambda}\pi=-\sqrt{\lambda}\cos\sqrt{\lambda}\pi$$

$$\Rightarrow \tan\sqrt{\lambda}\pi=-\sqrt{\lambda} \longrightarrow (3)$$

Thus eigenfunctions are $u_n=C_n\sin\sqrt{\lambda_n}x$ with eigenvalues λ_n where $n=0,\pm1,\pm2,\ldots$

λ_n satisfies the equation

$$\tan\sqrt{\lambda}\pi=-\sqrt{\lambda}$$

Question 4:

Find eigenvalues and eigen solutions of periodic SL system $u''+\lambda u=0$ with $u(0)=u(\pi)$, $u'(0)=u'(\pi)$.

Solution.

self proceed as example 2 (page 110).

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Square Integrable Function.

A function $f(x)$ is said to be square integrable with weight function $w(x) > 0$ over an interval $[a,b]$ if

$$\int_a^b w(x)\,|f(x)|^2\,dx < \infty$$

If $w(x) = 1$, then

$$\int_a^b |f(x)|^2\,dx < \infty$$

In this case the function $f(x)$ is simply called square integrable.

Lagrange's Identity.

Suppose $u(x)$ and $v(x)$ are two solutions of the SL equation, then the following identity must hold

$$uL(v) - vL(u) = \frac{d}{dx}\{p(x)(uv' - u'v)\}$$

which is called differential form of Lagrange's identity.

The integral form is given by

$$\int_a^b [uL(v) - vL(u)]\,dx = [p(x)\{uv' - u'v\}]_a^b$$

Derive Lagrange's Identity.

The ~~Lagrange's~~ SL operator is given as

$$L \equiv \frac{d}{dx}\left\{p(x)\frac{d}{dx}\right\} + q(x) \quad \text{——— (1)}$$

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Now $uL(v) - vL(u) = u\frac{d}{dx}\{p(x)\frac{dv}{dx}\} + uqv - v\frac{d}{dx}\{p(x)\frac{du}{dx}\} - vqu$

$= up(x)\frac{d^2v}{dx^2} + up'(x)\frac{dv}{dx} - vp(x)\frac{d^2u}{dx^2} - vp'(x)\frac{du}{dx}$

$= up(x)v'' + up'(x)v' - vp(x)u'' - vp'(x)u'$

$= p(x)(uv'' - vu'') + p'(x)(uv' - vu')$

$\Rightarrow uL(v) - vL(u) = \frac{d}{dx}[p(x)(uv' - vu')]$ ——— (2)

which is the differential form of the Lagrange's identity.

Now we define Wronskian of the functions $u(x)$ and $v(x)$ by $W(u,v)(x)$ as

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

(2) $\Rightarrow uL(v) - vL(u) = \frac{d}{dx}[p(x)W(u,v)(x)]$.

Taking integral on both sides from a to b, we get

$$\int_a^b [uL(v) - vL(u)]dx = \int_a^b \frac{d}{dx}[p(x)W(u,v)]dx$$

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$$\Rightarrow \int_a^b [uL(v) - vL(u)]dx = pW(u,v)\Big|_a^b$$

which is integral form of Lagrange's identity.

Remark.

If $W(u,v)(x) = 0$ for some specific value of $x \in [a,b]$. Then $W(u,v)(x) = 0$ on $[a,b]$.

Abel's Lemma.

If $u(x)$ and $v(x)$ are two solutions of an SL system corresponding to the eigenvalue λ on $[a,b]$.

Then $p(x)W(u,v)(x) =$ constant ; $x \in [a,b]$.

Proof.

Consider an SL system on $[a,b]$

$$Ly + \lambda r(x) y = 0 \quad \text{---------} \quad (1)$$

If $u(x)$ and $v(x)$ are two eigen solutions of equation (1), then,

$$Lu + \lambda r u = 0$$

and $Lv + \lambda r v = 0$

$$\Rightarrow Lu = -\lambda r u \longrightarrow (2)$$

and $Lv = -\lambda r v$ ---------- (3)

Multiplying (2) by v and (3) by u and then subtracting, we get,

$$uLv - vLu = -\lambda r uv + \lambda r uv$$

$$\Rightarrow uLv - vLu = 0 \longrightarrow (4)$$

The differential form of Lagrange's identity is given as

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$$uLv - vLu = \frac{d}{dx}\{p(uv' - vu')\} \longrightarrow (5)$$

Comparing (4) and (5), we get

$$\frac{d}{dx}\{p(uv' - vu')\} = 0$$

or $$\frac{d}{dx}\{pW(u,v)\} = 0$$

Integrating w.r.t. x, we get,

$$p(x)\,W(u,v)(x) = \text{Constant}, \quad x \in [a,b].$$

Theorem.

The eigenvalues of a regular SL system are simple.

OR There is only one (one and) linearly independent solution of a regular SL-system corresponding to an eigenvalue λ.

Proof.

Consider an SL system on $[a,b]$

$$Ly + \lambda r(x) y = 0 \quad \text{——} (1)$$

where $L = \frac{d}{dx}\{p(x)\frac{d}{dx}\} + q(x)$.

with end point conditions

$$\alpha y(a) + \alpha_1 y'(a) = 0 \quad \text{——} (2)$$

and $$\beta y(b) + \beta_1 y'(b) = 0 \quad \text{——} (3)$$

where $\alpha \neq 0, \alpha_1 \neq 0, \beta \neq 0$ and $\beta_1 \neq 0$.

Suppose that corresponding to an eigenvalue λ, there are two linearly independent solutions (say) u and v. Then

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$Lu + \lambda \rho u = 0$ ⟶ (4)

with $\alpha u(a) + \alpha_1 u'(a) = 0$ ⟶ (5)

and $\beta u(b) + \beta_1 u'(b) = 0$ ⟶ (6)

and

$Lv + \lambda \rho v = 0$ ⟶ (7)

with $\alpha v(a) + \alpha_1 v'(a) = 0$ ⟶ (8)

$\beta v(b) + \beta_1 v'(b) = 0$ ⟶ (9)

Multiplying (5) by $v'(a)$, (8) by $u'(a)$ and then subtracting, we get,

(5) ⇒ $\alpha u(a) v'(a) + \alpha_1 u'(a) v'(a) = 0$

(8) ⇒ $-\alpha v(a) u'(a) - \alpha_1 u'(a) v'(a) = 0$

$\alpha u(a) v'(a) - \alpha v(a) u'(a) = 0$

$\Rightarrow \alpha [u(a) v'(a) - v(a) u'(a)] = 0$

Since $\alpha \neq 0$, therefore,

$u(a) v'(a) - v(a) u'(a) = 0$

$\Rightarrow W(u, v)(a) = 0$

Using the property of Wronskian i.e. if Wronskian of two functions is zero at a point on $[a, b]$ then Wronskian of two functions is zero on $[a, b]$.

Thus $W(u, v)(x) = 0$, $\forall\ x \in [a, b]$.

$\Rightarrow$ u and v are linearly dependent.

Which is contradiction to our supposition.

Hence for an eigenvalue λ there exists only one linearly independent solution.

i.e. eigenvalues of a regular SL system are simple.

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Rayleigh Ritz Quotient Formula:

(Relationship b/w an eigenvalue λ and its corresponding eigenfunction $u(x)$).

$$\lambda = \frac{-[puu']_a^b + \int_a^b (pu'^2 - qu^2)\,dx}{\int_a^b \rho u^2\,dx}$$

Proof:

Consider an SL equation

$$\frac{d}{dx}\left\{p(x)\frac{du}{dx}\right\} + q(x)u + \lambda\rho(x)u = 0 \quad \text{---} (1)$$

or $(pu')' + qu + \lambda\rho u = 0$ ---(2)

Multiplying (2) by u, we get,

$$u(pu')' + qu^2 + \lambda\rho u^2 = 0$$

Integrating, we get, from a to b,

$$\int_a^b (u(pu')' + qu^2 + \lambda\rho u^2)\,dx = 0$$

$$\Rightarrow \int_a^b u(pu')'dx + \int_a^b qu^2dx + \int_a^b \lambda\rho u^2\,dx = 0$$

$$\Rightarrow \lambda\int_a^b \rho u^2dx = -\int_a^b u(pu')'dx - \int_a^b qu^2dx \longrightarrow (3)$$

Note that

$$\int_a^b u(pu')'dx = [u(pu')]_a^b - \int_a^b u'(pu')\,dx$$ Integrating by parts

$$\Rightarrow \int_a^b u(pu')dx = [u(pu')]_a^b - \int_a^b pu'^2\,dx$$

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Now equation (3) becomes

$$\lambda\int_a^b \rho u^2 dx = -[u(pu')]_a^b + \int_a^b pu'^2 dx - \int_a^b qu^2 dx$$

$$\Rightarrow \lambda = \frac{-[u(pu')]_a^b + \int_a^b (pu'^2 - qu^2)dx}{\int_a^b \rho u^2 dx}.$$

Note:

This result is not used to determine the eigenvalues, this is so because to evaluate λ we need to know the corresponding eigenfunction.

Example 1:

For the S-L eigenvalue problem

$$\frac{d^2u}{dx^2} + \lambda u = 0$$

with $u'(0) = 0$, $u'(l) = 0$

verify the following general results.

(i) There are infinite number of eigenvalues, with a smallest but no largest.

(ii) The nth eigenfunction has exactly $n-1$ zeros.

(iii) The eigenfunctions are orthogonal and form a complete set.

Solution.

Given that $\frac{d^2u}{dx^2} + \lambda u = 0$ ——— (1)

with $u'(0) = 0$; $u'(l) = 0$

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Therefore, the characteristic equation is

$$D^2 + \lambda = 0$$

$$\Rightarrow D^2 = -\lambda$$

$$\Rightarrow D = \pm i\sqrt{\lambda} \quad \text{(Complex)}$$

The general solution is

$$u(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x \quad \text{——} (2)$$

$$u'(x) = -C_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x + C_2 \lambda \cos\sqrt{\lambda}\, x$$

Applying end point conditions, we get,

$$u'(0) = 0 \Rightarrow -C_1(0) + C_2\sqrt{\lambda}\,(1) = 0$$

$$\Rightarrow \sqrt{\lambda}\, C_2 = 0$$

$$\Rightarrow C_2 = 0 \quad \because \lambda > 0.$$

$$u'(x) = -C_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x$$

$$u'(l) = 0 \Rightarrow -C_1\sqrt{\lambda} \sin\sqrt{\lambda}\, l = 0$$

Since $\lambda > 0$ and for non trivial solution, we must have,

$$\sin\sqrt{\lambda}\, l = 0$$

$$\Rightarrow \sqrt{\lambda}\, l = n\pi \quad ; \quad n = 0, 1, 2, \cdots$$

$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{l}$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{l^2} \quad ; \quad n = 0, 1, 2, 3, \cdots$$

Thus (2) $\Rightarrow u_n(x) = C_n \cos\sqrt{\lambda_n}\, x + 0 \quad \because C_2 = 0$

$$\Rightarrow u_n(x) = C_n \cos\frac{n\pi}{l} x \quad ; \quad n = 0, 1, 2 \cdots$$

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(i)

Clearly from eigenvalues $\lambda_n = \frac{n^2\pi^2}{\ell^2}$; $n = 0, 1, 2, \cdots$

for $n=0$, $\lambda_0 = 0$ is the smallest eigenvalue as all other eigenvalues ar +ive.

Also when $n \to \infty$, $\lambda_n \to \infty$ and the numbes of eigenvalues is infinite. Thus there are infinite eigenvalues with a smallest $\lambda_0 = 0$ but no largest.

(ii)

The nth eigenfunction is $U_n(x) = C_n \cos \frac{n\pi}{\ell} x$ where $n = 0, 1, 2, \cdots$

Here first eigen~~value~~ function is

$U_0(x) = C_0$ ∵ for non trivial solution $C_0 \neq 0$

Clearly it has no zero. (∵ there is no value of $x \in [0, \ell]$ for which $U_0(x) = 0$).

The 2nd eigenvalue is

$$U_1(x) = C_1 \cos \frac{\pi}{\ell} x$$

To find zeros of $U_1(x)$, we must have $U_1(x) = 0$

i.e. $\cos \frac{\pi}{\ell} x = 0$

$$\Rightarrow \frac{\pi x}{\ell} = (2m+1)\frac{\pi}{2} \; ; \; m \in \mathbb{Z}$$

$$\Rightarrow x = (2m+1)\frac{\ell}{2} \; ; \; m \in \mathbb{Z}$$

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For $m=0$, $x=\frac{l}{2} \in [0,l]$

For $m=1$, $x=\frac{3l}{2} \notin [0,l]$ and so on.

Thus here is only one zero which belongs to $[0,l]$.

$\Rightarrow$ Second eigenfunction has exactly one zero in $[0,l]$.

The third eigenfunction is

$$u_2(x) = C_2 \cos \frac{2\pi}{l} x.$$

For its zero, we must have $u_2(x)=0$.

i.e. $\cos \frac{2\pi}{l} x = 0$

$$\Rightarrow \frac{2\pi}{l} x = (2m+1)\frac{\pi}{2} \quad ; m \in \mathbb{Z}$$

$$\Rightarrow x = (2m+1) l/4 \quad ; \quad m \in \mathbb{Z}$$

For $m=0$; $x_0 = \frac{l}{4} \in [0,l]$

For $m=1$ $x_1 = 3l/4 \in [0,l]$

For $m=2$ $x_2 = \frac{5l}{4} \notin [0,l]$.

and so on.

Thus 3rd eigenfunction has exactly two zeros in $[0,l]$.

Continuing in this way, we can say that the nth eigenfunction has exactly n-1 zeros in $[0,l]$.

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(iii) Orthogonality.

Consider $U_m(x) = C_m \cos \frac{m\pi}{l} x$

and $U_n(x) = C_n \cos \frac{n\pi}{l} x$

be two eigenfunctions with $m \neq n$.

As $\int_0^l U_m(x) U_n(x) dx = \int_0^l C_m \cos \frac{m\pi}{l} x \; C_n \cos \frac{n\pi}{l} x \, dx$

$$= \int_0^l C_m C_n \cos \frac{m\pi}{l} x \cos \frac{n\pi}{l} x \, dx$$

$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

$$\Rightarrow \int_0^l U_m U_n dx = \frac{1}{2} \int_0^l C_m C_n \cos(m+n)\frac{\pi}{l} x + \cos(m-n)\frac{\pi}{l} x \, dx$$

$$= \frac{1}{2} C_m C_n \int_0^l \cos(m+n)\frac{\pi}{l} x + \cos(m-n)\frac{\pi}{l} x \, dx$$

$$= \frac{1}{2} C_m C_n (0)$$

$$\Rightarrow \int_0^l U_m U_n dx = 0.$$

Hence all eigenfunctions are orthogonal.

Complete.

Any function $U(x)$ can be written as

$$U(x) = \sum_{n=0}^{\infty} C_n U_n(x).$$

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Example 2:

Using the Rayleigh quotient, discuss the sign of the eigenvalue of the SL-problem,

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad u(0) = 0, \quad u(l) = 0.$$

Solution.

Here $p(x) = 1$, $q(x) = 0$, $s(x) = 1$, $a = 0$ and $b = l$.

The Rayleigh Ritz quotient formula is

$$\lambda = \frac{-[upu']_a^b + \int_a^b (pu'^2 - qu^2)\,dx}{\int_a^b s u^2 dx}$$

$$= \frac{-[u(b)p(b)u'(b) - u(a)p(a)u'(a)] + \int_a^b (pu'^2 - qu^2)dx}{\int_a^b s u^2 dx}$$

$$= \frac{-[u(l)p(l)u'(l) - u(0)p(0)u'(0)] + \int_0^l u'^2 dx}{\int_0^l u^2 dx}$$

$$\Rightarrow \lambda = \frac{\int_0^l u'^2 dx}{\int_0^l u^2 dx} \qquad \text{---------- (1)}$$

Since $u^2 > 0$ as u is an eigenfunction, so $u(x) \neq 0$ and $u'^2 \geq 0$, from (1) it is clear that $\lambda \geq 0$.

If $\lambda = 0$ is an eigenvalue then (1) implies

$$\int_0^l u'^2 dx = 0 \Rightarrow u'(x) = 0 \Rightarrow u(x) = \text{Constant} = c$$

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For the validity of boundary conditions C must be zero so $u=0$.

$\Rightarrow \lambda$ can not be an eigenvalue.

Hence $\lambda > 0$. (+ive).

Self-Adjoint Operator.

An operator A defined over a linear space of functions is called self-adjoint if

$$\langle u, Av \rangle = \langle Au, v \rangle$$

The integral form is given as

$$\int_a^b u(x)\, A\, v(x)\, dx = \int_a^b A\, u(x)\, v(x)\, dx$$

where u, v are real.

5/2018 Show that S-L operator is self-adjoint.

We know that the Lagrange's identity is

$$\int_a^b [uL(v) - vL(u)]\, dx = [p(x)\{uv' - u'v\}]_a^b \rightarrow (1)$$

Since the R.H.S of (1) is zero for regular or periodic S-L system.

Therefore, $\int_a^b [uL(v) - vL(u)]\, dx = 0$

$$\Rightarrow \langle u, Lv \rangle - \langle Lu, v \rangle = 0$$

$$\Rightarrow \langle u, Lv \rangle = \langle Lu, v \rangle$$

Hence SL operator is self-adjoint for regular or periodic system.

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Example 3:

Show that the following B.Cs yeild self-adjoint problem.

$u(a) = u(b)$ and $p(a)u'(a) = p(b)u'(b)$.

Solution.

The Lagrange's identity in integral form is given by

$$\int_a^b \{uL(v) - vL(u)\}dx = \{p(x)(uv' - u'v)\}_a^b$$

$$= p(b)\{u(b)v'(b) - u'(b)v(b)\} - p(a)\{u(a)v'(a) - u'(a)v(a)\}$$

$$= p(b)u(b)v'(b) - p(b)u'(b)v(b) - p(a)u(a)v'(a) + p(a)u'(a)v(a)$$

using given B.Cs, we get

$$\Rightarrow \int_a^b (uL(v) - vL(u))dx = p(b)u(b)v'(b) - p(b)u'(b)v(b) - p(b)u(b)v'(b) + p(b)u'(b)v(b)$$

$$= 0$$

Hence the operator is self-adjoint for these conditions.

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Exercises

Find all eigenvalues and eigenfunctions for the following SL-problems.

(1) 2014 $x^2u'' + xu' + \lambda u = 0$
$u(1) = 0, \quad u(b) = 0$

Solution.

Given that

$$x^2u'' + xu' + \lambda u = 0 \quad ; u(1)=0, u(b)=0$$

Therefore, $x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + \lambda u = 0$

$\Rightarrow x^2D^2u + xDu + \lambda u = 0 \quad \because D \equiv \frac{d}{dx}$.

which is a Cauchy Euler's equation.

To solve it we make the subsitution

$x = e^t, \Rightarrow t = \ln x$ &

$xD = \Delta$ and $x^2D^2 = \Delta(\Delta - 1)$.

$\Delta(\Delta-1)u + \Delta u + \lambda u = 0$

$(\Delta^2 - \Delta + \Delta + \lambda)u = 0$

$(\Delta^2 + \lambda)u = 0$

The characteric equation is

$$\Delta^2 + \lambda = 0$$

$\Rightarrow \Delta^2 = -\lambda$

$\Rightarrow \Delta = \pm i\sqrt{\lambda}$ (Complex).

The general solution is

$$u(t) = C_1 \sin\sqrt{\lambda}t + C_2 \cos\sqrt{\lambda}t$$

or $u(x) = C_1 \sin\sqrt{\lambda}\ln x + C_2 \cos\sqrt{\lambda}\ln x$

Applying boundary conditions, we get,

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$u(1)=0 \Rightarrow 0 = C_1(0) + C_2 \Rightarrow \boxed{C_2 = 0}$

So $u(x) = C_1 \sin\sqrt{\lambda}\ln x$ ——— (1)

$u(b)=0 \Rightarrow 0 = C_1 \sin\sqrt{\lambda}\ln b$

For non-trivial solution $C_1 \neq 0$.

$\Rightarrow \sin\sqrt{\lambda}\ln b = 0$

$\Rightarrow \sqrt{\lambda}\ln b = n\pi \quad ; n=0,1,2,\cdots$

$\Rightarrow \sqrt{\lambda} = \dfrac{n\pi}{\ln b} \quad ; n=0,1,2,\cdots$

$\Rightarrow \lambda_n = \left(\dfrac{n\pi}{\ln b}\right)^2 \quad ; n=0,1,2,\cdots$

are eigenvalues and

(1) $\Rightarrow u_n(x) = C_n \sin(\sqrt{\lambda_n}\ln x)$ are eigenfunctions.

(2) $y'' + \lambda y = 0 \ ; \ y'(-\pi) = 0, \ y'(\pi) = 0.$

Solution.

Given that $y'' + \lambda y = 0 \ ; \ y'(-\pi) = 0, \ y'(\pi) = 0$

The characteristic equation is

$D^2 + \lambda = 0$

$D^2 = -\lambda$

$D = \pm i\sqrt{\lambda}$ (Complex)

The general solution is

$y(x) = C_1 \sin\sqrt{\lambda}x + C_2 \cos\sqrt{\lambda}x$

For Applying given boundary conditions

$y'(x) = +C_1\sqrt{\lambda}\cos\sqrt{\lambda}x - C_2\sqrt{\lambda}\sin\sqrt{\lambda}x$

$y'(-\pi) = 0 \Rightarrow 0 = C_1\sqrt{\lambda}\cos\sqrt{\lambda}\pi + C_2\sqrt{\lambda}\sin\sqrt{\lambda}\pi$

$\Rightarrow \sqrt{\lambda}(C_1\cos\sqrt{\lambda}\pi + C_2\sin\sqrt{\lambda}\pi) = 0$

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Since $\lambda > 0$, so $\sqrt{\lambda} \neq 0$.

$\Rightarrow C_1 \cos\sqrt{\lambda}\pi + C_2 \sin\sqrt{\lambda}\pi = 0$ ——— (1)

$y'(\pi) = 0$

$\Rightarrow 0 = C_1\sqrt{\lambda}\cos\sqrt{\lambda}\pi - C_2\sqrt{\lambda}\sin\sqrt{\lambda}\pi$

$\Rightarrow \sqrt{\lambda}(C_1\cos\sqrt{\lambda}\pi - C_2\sin\sqrt{\lambda}\pi) = 0$

Since $\lambda > 0$, so $\sqrt{\lambda} \neq 0$.

$\Rightarrow C_1\cos\sqrt{\lambda}\pi - C_2\sin\sqrt{\lambda}\pi = 0$ ——— (2)

Adding (1) and (2), we get,

$C_1\cos\sqrt{\lambda}\pi + C_1\cos\sqrt{\lambda}\pi = 0$

$\Rightarrow 2C_1\cos\sqrt{\lambda}\pi = 0$

For non trivial solution $C_1 \neq 0$.

$\Rightarrow \cos\sqrt{\lambda}\pi = 0$

$\Rightarrow \sqrt{\lambda}\pi = (2n+1)\frac{\pi}{2}$; $n = 0, 1, 2, \cdots$

$\Rightarrow \sqrt{\lambda} = \frac{(2n+1)}{2}$

$\Rightarrow \lambda_n = \left[\frac{1}{2}(2n+1)\right]^2 = \left(n+\frac{1}{2}\right)^2$

Subtracting (2) from (1), we get,

$C_2\sin\sqrt{\lambda}\pi + C_2\sin\sqrt{\lambda}\pi = 0$

$\Rightarrow 2C_2\sin\sqrt{\lambda}\pi = 0$

For non trivial solution $C_2 \neq 0$.

$\Rightarrow \sin\sqrt{\lambda}\pi = 0$

$\Rightarrow \sqrt{\lambda}\pi = n\pi$; $n = 0, 1, 2, \cdots$

$\Rightarrow \sqrt{\lambda} = n$

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$\Rightarrow \lambda_n = n^2 \; ; \; n = 0, 1, 2, \cdots$

Thus, the corresponding eigenfunctions to $\lambda_n = n + \frac{1}{2}$ are

$y_n(x) = C_n \sin\sqrt{\lambda_n}\, x + C_2 \cos\sqrt{\lambda_n}\, x$

and the corresponding eigenfunctions to $\lambda_n = n$ are

$y_n(x) = C_n \sin\sqrt{n^2}x + C_n \cos\sqrt{n^2}x$

or $y_n(x) = C_n\left[\sin(n+\frac{1}{2})x + \cos(n+\frac{1}{2})x\right]$

and $y_n(x) = C_n[\sin(nx) + \cos(nx)]$.

(3) $y'' + \lambda^2 y \; , \; 0 \le x \le \pi$.

$y(0) + y(\pi) = 0 \; , \; y'(0) + y'(\pi) = 0$

Solution.

Given that $y'' + \lambda^2 y, \; 0 \le x \le \pi$

with $y(0) + y(\pi) = 0, \; y'(0) + y'(\pi) = 0$

Therefore, characteristic equation is

$D^2 + \lambda = 0$

$\Rightarrow D = \pm i\sqrt{\lambda} \quad (\lambda > 0)$.

The general solution is

$y(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$ — (1)

$y(0) + y(\pi) = 0$

$\Rightarrow C_1(1) + C_2(0) + C_1 \cos\sqrt{\lambda}\,\pi + C_2 \sin\sqrt{\lambda}\,\pi = 0$

$\Rightarrow C_1 + C_1 \cos\sqrt{\lambda}\,\pi + C_2 \sin\sqrt{\lambda}\,\pi = 0$

$\Rightarrow C_1(1 + \cos\sqrt{\lambda}\,\pi) + C_2 \sin\sqrt{\lambda}\,\pi = 0$ — (2)

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① $\Rightarrow y'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

$y'(0) + y'(\pi) = 0$

$-C_1(0) + C_2\sqrt{\lambda}(1) - C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$

$\Rightarrow C_2\sqrt{\lambda} - C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$

$\Rightarrow C_2\sqrt{\lambda}(1+\cos\sqrt{\lambda}\pi) - C_1\sqrt{\lambda}\sin\sqrt{\lambda}\pi = 0$

$\Rightarrow \sqrt{\lambda}[C_2(1+\cos\sqrt{\lambda}\pi) - C_1\sin\sqrt{\lambda}\pi] = 0$

$\Rightarrow C_2(1+\cos\sqrt{\lambda}\pi) - C_1\sin\sqrt{\lambda}\pi = 0$ — ②

Multiplying ② by $1+\cos\sqrt{\lambda}\pi$ and ③ by $\sin\sqrt{\lambda}\pi$ and then subtracting we get,

$C_1(1+\cos\sqrt{\lambda}\pi)^2 + C_2(1+\cos\sqrt{\lambda}\pi)\sin\sqrt{\lambda}\pi + C_1\sin^2\sqrt{\lambda}\pi - C_2(1+\cos\sqrt{\lambda}\pi)\sin\sqrt{\lambda}\pi = 0$

$\Rightarrow C_1[1+\cos^2\sqrt{\lambda}\pi + 2\cos\sqrt{\lambda}\pi + \sin^2\sqrt{\lambda}\pi] = 0$

$\Rightarrow C_1(2+2\cos\sqrt{\lambda}\pi) = 0$

$\Rightarrow 2C_1(1+\cos\sqrt{\lambda}\pi) = 0$

For non trivial solution $C_1 \neq 0$

$\Rightarrow 1+\cos\sqrt{\lambda}\pi = 0$

$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\Rightarrow 2\cos^2\frac{\sqrt{\lambda}}{2}\pi = 0$

$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$

$\Rightarrow \cos^2\frac{\sqrt{\lambda}}{2}\pi = 0$

$\Rightarrow \cos\frac{\sqrt{\lambda}}{2}\pi = 0$

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$\Rightarrow \frac{\sqrt{\lambda}}{2}\pi = (2n+1)\frac{\pi}{2}$; $n=0,1,2,\cdots$

$\Rightarrow \sqrt{\lambda} = 2n+1$; $n=0,1,2,\cdots$

$\Rightarrow \lambda_n = (2n+1)^2$; $n=0,1,2,\cdots$

Multiplying ② by $\sin\sqrt{\lambda}\pi$ and ② by $1+\cos\sqrt{\lambda}\pi$ and then adding, we get,

$C_1(1+\cos\sqrt{\lambda}\pi)\sin\sqrt{\lambda}\pi + C_2\sin^2\sqrt{\lambda}\pi$

$- C_1(1+\cos\sqrt{\lambda}\pi)\sin\sqrt{\lambda}\pi + C_2(1+\cos\sqrt{\lambda}\pi)^2 = 0$

$\Rightarrow C_2[\sin^2\sqrt{\lambda}\pi + 1 + \cos^2\sqrt{\lambda}\pi + 2\cos\sqrt{\lambda}\pi] = 0$

$\Rightarrow 2C_2[1+\cos\sqrt{\lambda}\pi] = 0$

For non trivial solution we must have $C_2 \neq 0$

$\Rightarrow 1+\cos\sqrt{\lambda}\pi = 0$

$2\cos^2\frac{\sqrt{\lambda}}{2}\pi = 0$

$\Rightarrow \lambda_n = (2n+1)^2$; $n=0,1,2,\cdots$

Thus by ① corresponding to the eigenvalues $\lambda_n = (2n+1)^2$ the eigenfunctions are

$y_n(x) = C_n[\cos(2n+1)x + \sin(2n+1)x]$.

(4) 5/2018 $(e^x u')' + e^x\lambda^2 u = 0$, $0<x<a$.

$u(0) = 0$, $u(a) = 0$.

Solution.

Given that $(e^x u')' + e^x\lambda^2 u = 0$

$u(0) = 0$, $u(a) = 0$

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Therefore, $e^x u'' + e^x u' + e^x \lambda^2 u = 0$

or $e^x\left[\frac{d^2u}{dx^2} + \frac{du}{dx} + \lambda^2 u\right] = 0$

$\Rightarrow D^2u + Du + \lambda^2 u = 0 \quad \because e^x \neq 0.$

$\Rightarrow (D^2 + D + \lambda^2)u = 0$

The auxiliary equation is

$D^2 + D + \lambda^2 = 0$

by using quadratic formula,

$$D = \frac{-1 \pm \sqrt{1 - 4\lambda^2}}{2} = \frac{-1 \pm \sqrt{4\lambda^2 - 1}\, i}{2}$$

$$\Rightarrow D = -\frac{1}{2} \pm \frac{\sqrt{4\lambda^2 - 1}}{2} i \quad \text{(Complex)}$$

The general solution is

$$u(x) = \left[C_1 \cos\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)x + C_2 \sin\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)x\right] e^{-x/2}$$

Applying given boundary condition, we get,

$u(0) = 0 \Rightarrow 0 = [C_1 \cos 0^\circ + C_2(0)]\, e^0$

$$\Rightarrow \boxed{C_1 = 0}$$

$$u(a) = 0 \Rightarrow 0 = \left[C_1 \cos\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)a + C_2 \sin\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)a\right] e^{-a/2}$$

$$\Rightarrow C_2 \sin\left(\frac{\sqrt{4\lambda^2 - 1}}{2}\right)a = 0 \quad \because e^{-a/2} \neq 0$$

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$\Rightarrow \sin\frac{\sqrt{4\lambda^2-1}}{2}a = 0$ ∵ For non trivial solution $C_2 \neq 0$.

$\Rightarrow \frac{\sqrt{4\lambda^2-1}}{2}a = n\pi$; $n = 1, 2, 3, \cdots$

$\Rightarrow \sqrt{4\lambda^2-1} = \frac{2n\pi}{a}$

$\Rightarrow 4\lambda^2 - 1 = \left(\frac{2n\pi}{a}\right)^2 \Rightarrow 4\lambda^2 = 1 + \frac{4n^2\pi^2}{a^2}$

$\Rightarrow \lambda_n^2 = \left(\frac{n\pi}{a}\right)^2 + \frac{1}{4}$; $n = 1, 2, 3, \cdots$

Thus the required eigenvectors are

$u_n(x) = e^{-x/2} C_n \sin\frac{n\pi}{a}x$.

(5) 2018 1A $y'' + (1+\lambda)y = 0$; $y(0) = 0$, $y(\pi) = 0$

Solution.

Given that $y'' + (1+\lambda)y = 0$; $y(0) = 0$, $y(\pi) = 0$

Therefore, $(D^2 + 1 + \lambda)y = 0$

The characteristic equation is

$D^2 + (1+\lambda) = 0$

$D^2 = -(1+\lambda)$ ——— (1)

Case I. $1+\lambda = 0$

$D^2 = 0$; $D = 0, 0$ (real repeated)

The general solution is

$y(x) = (C_1 + C_2 x)e^{0x} = C_1 + C_2 x$

Applying given boundary conditions, we get,

$y(0) = 0 \Rightarrow \boxed{0 = C_1}$

$y(\pi) = 0$ $0 = C_2\pi \Rightarrow C_2 = 0$; $\pi \neq 0$

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So $y = 0$ (trivial solution).

Hence in this case we have no eigenvalue and no eigensolutions.

Case II: $1 + \lambda < 0$

① $\Rightarrow D^2 = 1 + \lambda$

$\Rightarrow D = \pm\sqrt{1+\lambda}$ (real, distinct)

The general solution is

$$y(x) = C_1 e^{\sqrt{1+\lambda}\,x} + C_2 e^{-\sqrt{1+\lambda}\,x}$$

Applying the given boundary conditions, we have,

$y(0) = 0 \Rightarrow 0 = C_1 + C_2$

$\Rightarrow \boxed{C_1 = -C_2}$

$$y(\pi) = 0 \Rightarrow 0 = C_1 e^{\sqrt{1+\lambda}\,\pi} + C_2 e^{-\sqrt{1+\lambda}\,\pi}$$

$$\Rightarrow C_1 e^{\sqrt{1+\lambda}\,\pi} - C_1 e^{-\sqrt{1+\lambda}\,\pi} = 0$$

$$\Rightarrow C_1\left[e^{\sqrt{1+\lambda}\,\pi} - e^{-\sqrt{1+\lambda}\,\pi}\right] = 0$$

Since $e^{\sqrt{1+\lambda}\,\pi} - e^{-\sqrt{1+\lambda}\,\pi} \neq 0$.

So $C_1 = 0$

$\Rightarrow C_2 = 0$

Thus $y = 0$ (trivial solution).

Hence in this case we have no eigenvalue and no eigensolutions.

Case III: $1 + \lambda > 0$

① $\Rightarrow D^2 = -(1+\lambda)$

$D = \pm i\sqrt{1+\lambda}$

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The general solution is

$y(x) = C_1 \cos\sqrt{1+\lambda}\, x + C_2 \sin\sqrt{1+\lambda}\, x$

Applying given BCs, we get,

$y(0) = 0 \Rightarrow \boxed{0 = C_1}$

So $y(x) = C_2 \sin\sqrt{1+\lambda}\, x$ ——— (2)

$y(\pi) = 0 \Rightarrow 0 = C_2 \sin\sqrt{1+\lambda}\,\pi$

For non trivial solution, we must have, $C_2 \neq 0$.

$\Rightarrow \sin\sqrt{1+\lambda}\,\pi = 0$

$\Rightarrow \sqrt{1+\lambda}\,\pi = n\pi \quad ; n = 1, 2, \cdots$

$\Rightarrow \sqrt{1+\lambda} = n$

$\Rightarrow \lambda_n = n^2 - 1 \quad ; n = 1, 2, 3, \cdots$

The eigenfunctions corresponding to these eigenvalues are

(2) $\Rightarrow y_n(x) = C_n \sin nx$.

(6) $y'' + 2y' + (1+\lambda)y = 0$

$y'(0) = 0$, $y'(\pi) = 0$.

Solution.

Given that $y'' + 2y' + (1-\lambda)y = 0$

with $y'(0) = 0$, $y'(\pi) = 0$

Therefore, $(D^2 + 2D + 1 + \lambda)y = 0$

The characteristic equation is

$D^2 + 2D + (1+\lambda) = 0$

Using quadratic formula, we get,

$D = \dfrac{-2 \pm \sqrt{4 - 4(1+\lambda)}}{2} = \dfrac{-2 \pm \sqrt{-4\lambda}}{2}$

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$$\Rightarrow D = \frac{-2 \pm 2\sqrt{\lambda}\, i}{2} = -1 \pm \sqrt{\lambda}\, i$$

$\Rightarrow D = -1 \pm \sqrt{\lambda}\, i$ —— ① (~~Complex~~)

Case I: $\lambda = 0$

① $\Rightarrow D = -1$ (real)

The general solution is

$y(x) = C_1 e^{-x}$

$y'(x) = -C_1 e^{-x}$

$y'(0) = 0 \Rightarrow C_1 = 0$

$y'(\pi) = 0 \Rightarrow C_1 = 0$

So $y = 0$.

Hence in this case we have no eigenvalues and no eigensolutions.

Case II: $\lambda > 0$.

① $\Rightarrow D = -1 \pm \sqrt{\lambda}\, i$ (Complex)

The general solution is

$y(x) = [C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x]\, e^{-x}$.

$y'(x) = (-C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x)e^{-x}$

$\quad -e^{-x}(C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x)$

$y'(0) = 0$

$\Rightarrow 0 = C_2\sqrt{\lambda} - C_1$

$\Rightarrow C_1 = \sqrt{\lambda}\, C_2$ —— ②

$y'(\pi) = 0$

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$$\Rightarrow \left(-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\bar{x} + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\bar{x}\right)e^{-\bar{x}}$$

$$-e^{-\bar{x}}\left(C_1\cos\sqrt{\lambda}\bar{x} + C_2\sin\sqrt{\lambda}\bar{x}\right) = 0.$$

$$\Rightarrow -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\bar{x} + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} - C_1\cos\sqrt{\lambda}\bar{x} - C_2\sin\sqrt{\lambda}\bar{x} = 0 \quad \because e^{-\bar{x}} \neq 0.$$

$$\Rightarrow -C_1\left(\sqrt{\lambda}\sin\sqrt{\lambda}\bar{x} + \cos\sqrt{\lambda}\bar{x}\right) + C_2\left(\sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} - \sin\sqrt{\lambda}\bar{x}\right) = 0 \quad \text{---(3)}$$

From (2) and (3), we can write,

$$\begin{bmatrix} -1 & \sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\bar{x} - \cos\sqrt{\lambda}\bar{x} & \sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} - \sin\sqrt{\lambda}\bar{x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \vec{0}$$

Since $C_1 = 0 = C_2$, give trivial solution.

For non trivial solution, we must have,

$$\begin{vmatrix} -1 & \sqrt{\lambda} \\ -\sqrt{\lambda}\sin\sqrt{\lambda}\bar{x} - \cos\sqrt{\lambda}\bar{x} & \sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} - \sin\sqrt{\lambda}\bar{x} \end{vmatrix} = 0$$

$$\Rightarrow -\sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} + \sin\sqrt{\lambda}\bar{x} + (\sqrt{\lambda})^2\sin\sqrt{\lambda}\bar{x} + \sqrt{\lambda}\cos\sqrt{\lambda}\bar{x} = 0$$

$$\Rightarrow \sin\sqrt{\lambda}\bar{x} + \lambda\sin\sqrt{\lambda}\bar{x} = 0$$

$$\Rightarrow (1+\lambda)\sin\sqrt{\lambda}\bar{x} = 0.$$

Since $\lambda > 0$, so $1+\lambda \neq 0$.

$$\Rightarrow \sin\sqrt{\lambda}\bar{x} = 0$$

$$\Rightarrow \sqrt{\lambda}\bar{x} = n\bar{x} \quad ; n = 1, 2, 3, \cdots$$

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$\Rightarrow \sqrt{\lambda} = n$

$\Rightarrow \lambda_n = n^2 \quad ; \quad n = 1, 2, 3.$

The eigen solutions corresponding to these eigenvalues are

$y_n(x) = C_n[\cos nx + \sin nx]e^{-x}.$

⑦ 2013 2015 $u'' + \lambda u = 0 \; ; \; u(0) = 0, \; u'(\frac{\pi}{2}) = 0.$

Solution.

Given that $u'' + \lambda u = 0$ ——— ①

with $u(0) = 0, \; u'(\frac{\pi}{2}) = 0$

The characteristic equation is

$D^2 + \lambda = 0$

$\Rightarrow D^2 = -\lambda$

$\Rightarrow D = -i\sqrt{\lambda}$ (Complex)

The general solution is

$u(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$ ——— ②

Applying boundary conditions, we have,

$u(0) = 0 \Rightarrow \boxed{0 = C_1}$

② $\Rightarrow u(x) = C_2 \sin\sqrt{\lambda}x$ ——— ③

$u'(x) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x - C_1\sqrt{\lambda}\sin\sqrt{\lambda}x$

$u'(\frac{\pi}{2}) = 0 \Rightarrow 0 = C_2\sqrt{\lambda}\cos\sqrt{\lambda}\frac{\pi}{2} - C_1\sqrt{\lambda}\sin\sqrt{\lambda}\frac{\pi}{2}$

$\Rightarrow C_2\sqrt{\lambda}\cos\sqrt{\lambda}\frac{\pi}{2} = 0 \quad \because C_1 = 0$

For non trivial solution, we must have,

$C_2 \neq 0$

$\Rightarrow \cos\sqrt{\lambda}\frac{\pi}{2} = 0 \quad \because \sqrt{\lambda} \neq 0 \text{ as } \lambda > 0.$

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$\Rightarrow \sqrt{\lambda}\,\frac{\pi}{2} = (2n+1)\frac{\pi}{2}$ $\quad \because n = 0, 1, 2, \cdots$

$\Rightarrow \sqrt{\lambda} = 2n+1$

$\Rightarrow \lambda_n = (2n+1)^2 \quad ; n = 0, 1, 2, \cdots$

are eigenvalues and corresponding eigenfunctions are

③ $\Rightarrow u_n(x) = C_n \sin(2n+1)x$. Ans.

Other two cases i.e. $\lambda = 0$ and $\lambda < 0$ can be solved similarly but theis solutions are trivial

Q1: Solve

$$\frac{d}{dx}\left[(x^2+1)\frac{dy}{dx}\right] + \frac{\lambda}{x^2+1}y = 0$$

$$y(0) = 0, \quad y(1) = 0$$

Q2: $(1+x)^2 y'' + y = 0 \quad ; x_0 = 0$

Q3: $y'' + 2xy' + y = 0.$

Q4: $y'' + 2xy = 0 \quad ; x_0 = 0$

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One Dimensional Heat Equation.

Conduction of heat.

The mode in which heat transfer from one part of body to another through the interaction of molecules of that body is called conduction.

Convection of heat.

The mode in which heat transfer from one part of body to another through the actual movement of molecules of that body is called convection.

Radiation.

The mode in which heat transfer from one place to another through waves is called radiation.

Specific heat of substance.

The specific heat of a substance is defined as the quantity of heat required to raise the temperature of one gram of substance by one degree centigrade.
It is denoted by c.

Thermal conductivity.

The quantity of heat flowing per second across a plate of unit area and unit thickness, when the temperature difference between opposite sides is 1 degree centigrade.
It is denoted by K.

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Rate of flow of heat.

The quantity of heat entering per second through unit area perpendicular to the direction of flow is called rate of flow of heat. It is denoted by $q(x,t)$.

Diffusivity.

The measure of ability of a material to respond to change in its thermal environment is called diffusivity. It is denoted by K.

Law of conservation of heat energy.

Law of conservation of heat energy states that the quantity of heat which enters a region plus what is generated inside is equal to quantity of heat which leaves plus the quantity of heat stored.

Fourier's Law.

This law states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area, at right angle to that gradient through which heat flows.

i.e $q(x,t) = -K \nabla u$.

Question. 2015

Derive one dimensional heat equation. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K} \frac{\partial u}{\partial t}$

Solution.

Consider a uniform rod of length l on the x-axis

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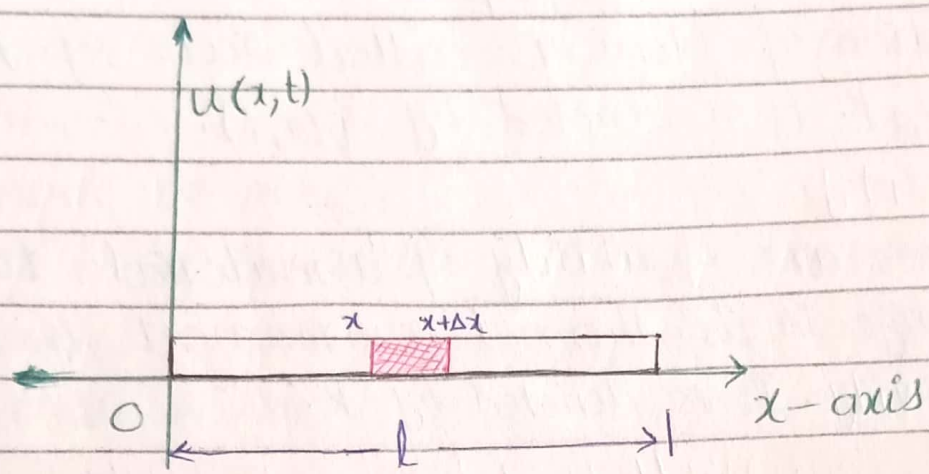
from $x=0$ to $x=l$. By uniform rod, we mean the density ρ, specific heat c and cross sectional area A are all constant.



Let us consides an arbitrary thin slice of the rod between x and $x+\Delta x$. The slice is so thin that the temperature throughout the slice is $u(x,t)$.

Thus heat energy of this segment is given as

Heat energy $= c\rho A \Delta x\, u(x,t)$.

Applying law of conservation of heat (energy) to an element of the rod bounded between x and $x+\Delta x$, we have,

$q(x,t)A + \ell A \Delta x = q(x+\Delta x, t)A +$ heat stored per second.

where $q(x,t)$ denotes the rate of flow of heat.

$$q(x,t)A + \ell A \Delta x = q(x+\Delta x, t)A + c\rho A \Delta x \frac{\partial}{\partial t} u(x,t)$$

$$\Rightarrow q(x,t) + \ell \Delta x = q(x+\Delta x, t) + c\rho \Delta x \frac{\partial}{\partial t} u(x,t).$$

as A is constant.

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$\Rightarrow q(x,t) + s\Delta x - q(x+\Delta x,t) = c\rho\Delta x \frac{\partial}{\partial t} u(x,t)$

Dividing both sides by Δx, we get,

$$\frac{q(x,t) - q(x+\Delta x,t)}{\Delta x} + s = c\rho \frac{\partial}{\partial t} u(x,t).$$

or $$-\frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} + s = c\rho \frac{\partial}{\partial t} u(x,t).$$

Taking Limit $\Delta x \to 0$, we get,

$$-\lim_{\Delta x \to 0} \frac{q(x+\Delta x,t) - q(x,t)}{\Delta x} + s = c\rho \frac{\partial}{\partial t} u(x,t).$$

$$\Rightarrow -\frac{\partial q(x,t)}{\partial x} + s = c\rho \frac{\partial}{\partial t} u(x,t). \longrightarrow (1)$$

which contains two dependent variables q, u.

Fourier's law in one dimension is

$$q(x,t) = -K \frac{\partial u}{\partial x} \longrightarrow (2)$$

Substituting (2) in (1), we get,

$$-\frac{\partial}{\partial x}\left(-K\frac{\partial u}{\partial x}\right) + s = c\rho \frac{\partial}{\partial t} u(x,t).$$

To obtain standard form of the heat equation, we suppose that $s = 0$ and K is constant, then,

$$K \frac{\partial^2 u}{\partial x^2} = c\rho \frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{c\rho}{K} \frac{\partial u}{\partial t}$$

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$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \longrightarrow (3)$$

where k is diffusivity of the material.
Thus (3) is required one dimensional heat equation.

Example 1. 2016,
Interpret and solve the following problem.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \quad t > 0. \quad \text{---} (1)$$

$$u(0,t) = T_0, \quad u(a,t) = T_1, \quad t > 0. \quad \text{---} (2)$$

$$u(x,0) = f(x), \quad 0 < x < a. \quad \text{---} (3)$$

Solution.
First we interpret the problem physically as follows:
The differential equation describes the flow of heat in a rod of length a, which is insulated along the sides and whose endpoints are maintained at temperatures at T_0 and T_1. The original temperature distribution is given by $f(x)$.
Now we solve the problem as follows.
First we determine steady state solution defined by

$$V(x) = \lim_{t\to\infty} u(x,t)$$

$$(1) \Rightarrow \lim_{t\to\infty} \frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \lim_{t\to\infty} u(x,t)$$

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$$\Rightarrow \frac{\partial^2 V(x)}{\partial x^2} = \frac{1}{K}\frac{\partial V(x)}{\partial t}$$

نوٹ:- V is function of x so $\frac{\partial V}{\partial t} = 0$

$$\Rightarrow \frac{d^2 V(x)}{dx^2} = 0$$

$$\Rightarrow D^2 V(x) = 0 \qquad \because D^2 = \frac{d^2}{dx^2}$$

The characteristic equation is

$$D^2 = 0 \Rightarrow D = 0, 0 \quad \text{(real, repeated)}$$

The general solution is

$$V(x) = C_1 + C_2 x \qquad \text{——— (4)}$$

② and ③ yeild,

نوٹ:- Separation of Variable method ← homogeneous Problem

② $\Rightarrow \lim_{t\to\infty} u(0,t) = T_0$

$\Rightarrow V(0) = T_0$

and $\lim_{t\to\infty} u(a,t) = T_1 \Rightarrow V(a) = T_1$

③ $\Rightarrow \lim_{t\to\infty} u(x,0) = f(x) \Rightarrow V(x) = f(x)$

Thus equation (4) implies

$V(0) = C_1 \Rightarrow \boxed{T_0 = C_1}$

$V(a) = C_1 + C_2 a \Rightarrow T_1 = C_1 + C_2 a$

$\Rightarrow T_1 = T_0 + C_2 a$

$\Rightarrow T_1 - T_0 = C_2 a \Rightarrow \boxed{C_2 = \frac{T_1 - T_0}{a}}$

Thus steady state solution is

$$V(x) = T_0 + \left(\frac{T_1 - T_0}{a}\right)x \qquad \text{——— (5)}$$

Now we find transient solution $w(x,t)$ which is given by

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$$w(x,t) = u(x,t) - v(x)$$

$$\Rightarrow u(x,t) = w(x,t) + v(x).$$

$$① \Rightarrow \frac{\partial^2}{\partial x^2}[w(x,t)+v(x)] = \frac{1}{K}\frac{\partial}{\partial t}[w(x,t)+v(x)]$$

$$\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} + \cancelto{0}{\frac{\partial^2 v(x)}{\partial x^2}} = \frac{1}{K}\left(\frac{\partial w(x,t)}{\partial t} + \cancelto{0}{\frac{\partial v(x)}{\partial t}}\right)$$

double ⑤ کا derivative w.r.t x ہوگا zero

$$\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{1}{K}\frac{\partial w(x,t)}{\partial t} \longrightarrow ⑥$$

From $u(x,t) = w(x,t) + v(x)$ ② and ③ yeild,

$$② \Rightarrow w(0,t) + v(0) = T_0$$

$$\Rightarrow w(0,t) + T_0 = T_0$$

$$\Rightarrow w(0,t) = 0 \quad \text{———} \quad ⑦$$

and $w(a,t) + v(a) = T_1$

$$\Rightarrow w(a,t) + T_1 = T_1$$

$$\Rightarrow w(a,t) = 0 \quad \text{———} \quad ⑧$$

$$③ \Rightarrow w(x,0) + v(x) = f(x)$$

نوٹ:

$$\Rightarrow w(x,0) = f(x) - v(x)$$

$$\Rightarrow w(x,0) = g(x) \quad \text{(say)} \longrightarrow ⑨$$

$f(x)$ اور $v(x)$ دونوں functions of x ہیں گو ان کا difference بھی function of x ہوگا

Here for $w(x,t)$, the boundary conditions are homogeneous so we can solve this problem ⑥ to ⑨ by separation of variables method.

Let $w(x,t) = X(x)\,T(t)$

Then ⑥ gives us

$$\frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right) = \frac{1}{K}\frac{\partial}{\partial t}\left(X(x)T(t)\right). \quad 0<x<a,\ t>0$$

$$\Rightarrow T(t)\frac{\partial^2 X}{\partial x^2} = \frac{1}{K}\left(X(x)\frac{\partial T}{\partial t}\right).$$

$$\Rightarrow T\frac{d^2X}{dx^2} = \frac{1}{K}X\frac{dT}{dt}, \quad 0<x<a,\ t>0.$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{KT}\frac{dT}{dt} \quad \text{———} \quad (10)$$

Since L.H.S is a function of x only and R.H.S is a function of t only, both sides are equal only if they are equal to some constant (say) $-\lambda^2$ where $\lambda \in \mathbb{R}\setminus\{0\}$.

$$(10) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \quad \text{———} \quad (11)$$

and $$\frac{1}{KT}\frac{dT}{dt} = -\lambda^2 \quad \text{———} \quad (12)$$

$$(7) \Rightarrow W(0,t) = X(0)T(0) = 0$$

$$\Rightarrow X(0) = 0 \quad \text{———} \quad (13)$$

نوٹ:- اگر $T(t)$ کو zero کریں ہمارا solution trivial بن جاتا ہے

$$(8) \Rightarrow W(a,t) = X(a)T(a) = 0$$

$$\Rightarrow X(a) = 0 \quad \text{———} \quad (14)$$

Now,

$$(11) \Rightarrow \frac{1}{X}D^2X + \lambda^2 = 0$$

$$\Rightarrow D^2X + \lambda^2 X = 0$$

The auxilary equation is

$$D^2 + \lambda^2 = 0 \quad \Rightarrow D^2 = -\lambda^2$$

$$\Rightarrow D = i\lambda \quad \text{(Complex)}$$

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The general solution is

$$X(x) = C_3 \cos\lambda x + C_4 \sin\lambda x$$

$X(0) = 0 \Rightarrow \boxed{C_3 = 0} \Rightarrow X(x) = C_4 \sin\lambda x$

$X(a) = 0 \Rightarrow 0 = C_4 \sin\lambda a$

For non trivial solution we choose $C_4 \neq 0$.

$\Rightarrow \sin\lambda a = 0 \Rightarrow \lambda a = n\pi \quad ; n = 0, \pm1, \pm2, \cdots$

$\Rightarrow \lambda = \frac{n\pi}{a} \quad ; n = 0, \pm1, \pm2, \cdots$

$\Rightarrow \lambda_n = \frac{n\pi}{a} \quad ; n = \pm1, \pm2, \cdots$

Here we have ignored $n=0$ because for $n=0$ we have trivial solution.

Thus the eigensolutions are

$$X_n(x) = C_n \sin\lambda_n x$$

where $\lambda_n = \frac{n\pi}{a} \quad ; n = 1, 2, 3, \cdots$

We ignored negative values of n because they donot give linearly independent solutions.

(12) $\Rightarrow \frac{1}{KT}\frac{dT}{dt} = -\lambda^2$

$\Rightarrow \frac{dT}{dt} = -\lambda^2 KT$

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$\Rightarrow d\bar{T} = -\lambda^2 k \bar{T} dt \quad \Rightarrow \frac{d\bar{T}}{\bar{T}} = -\lambda^2 k dt$

Integrating, we get

$\ln \bar{T} = -\lambda^2 k t + \ln C_5$

$\Rightarrow \ln \bar{T} = \ln e^{-\lambda^2 k t} + \ln C_5 = \ln C_5 e^{-\lambda^2 k t}$

$\Rightarrow \ln \bar{T} = \ln C_5 e^{-\lambda^2 k t}$

$\Rightarrow \bar{T} = C_5 e^{-\lambda^2 k t}$ (taking anti ln we get)

Thus $\bar{T}_n(t) = C_n^* e^{-\lambda^2 k t}$

Thus $W_n(x,t) = X_n(x) \bar{T}_n(t)$ implies

$W_n(x,t) = (C_n \sin \lambda_n x) C_n^* e^{-\lambda^2 k t}$

$= C_n C_n^* \sin \lambda_n x \, e^{-\lambda^2 k t}$

where $\lambda_n = \frac{n\pi}{a}$; $n = 1, 2, 3, \ldots$

By principle of superposition, we have,

$$W(x,t) = \sum_{n=1}^{\infty} C_n C_n^* \sin \lambda_n x \, e^{-\lambda^2 k t}$$

$$= \sum_{n=1}^{\infty} b_n \sin \lambda_n x \, e^{-\lambda^2 k t} \quad \because b_n = C_n C_n^*$$

$$\Rightarrow W(x,0) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x$$

$$\textcircled{9} \Rightarrow g(x) = \sum_{n=1}^{\infty} b_n \sin \lambda_n x$$

This is a Fourier Sine series of $g(x)$.

(next see page 82 book)

Example 2. A/2018, 2011,

Find the steady state solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K} \frac{\partial u}{\partial t}, \quad 0 < x < a, \; t > 0$$

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Subject to

$u_x(0,t) = 0$, $u_x(a,t) = 0$; $t > 0$

$u(x,0) = f(x)$, $0 < x < a$.

Discuss uniqueness of the solution.

Solution.

Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0 < x < a, \ t > 0 \quad \text{——} (1)$$

$$u_x(0,t) = 0, \ u_x(a,t) = 0, \ t > 0 \quad \text{——} (2)$$

$$u(x,0) = f(x), \ 0 < x < a \quad \text{——} (3)$$

Taking limit of the given problem for steady state solution and $\lim_{t\to\infty} u(x,t) = v(x)$ where $v(x)$ is steady state solution.

$$(1) \Rightarrow \lim_{t\to\infty} \frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \lim_{t\to\infty} \frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v(x)}{\partial x^2} = \frac{1}{k}\frac{\partial v}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v(x)}{\partial x^2} = 0 \qquad \because \frac{\partial v(x)}{\partial t} = 0$$

$$\Rightarrow \frac{d^2 v(x)}{dx^2} = 0 \Rightarrow D^2 v(x) = 0$$

The characteristic equation is

$D^2 = 0$; $D = 0, 0$ (real, repeated)

The general solution is

$$v(x) = C_1 + C_2 x \quad \text{——} (4)$$

(2) and (3) imply that

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$$\lim_{t\to\infty} \frac{\partial u(x,t)}{\partial x}\bigg|_{x=0} = 0 \Rightarrow \frac{\partial V(x)}{\partial x} = 0$$

$$\Rightarrow \frac{dV(x)}{dx}\bigg|_{x=0} = 0 \quad \text{——— (5)}$$

and $\lim_{t\to\infty} \frac{\partial u(x,t)}{\partial x}\bigg|_{x=a} = 0 \Rightarrow \frac{\partial V(x)}{\partial x}\bigg|_{x=a} = 0$

$$\Rightarrow \frac{dV(x)}{dx}\bigg|_{x=a} = 0 \quad \text{——— (6)}$$

(3) $\Rightarrow \lim_{t\to\infty} u(x,0) = f(x) \Rightarrow V(x) = f(x)$ ——— (6)

(4) $\Rightarrow \frac{dV}{dx} = C_2$ ——— (7)

using (5) in (7), we have, $\boxed{C_2 = 0}$

(4) $\Rightarrow V(x) = C_1$

$\Rightarrow \frac{dV(x)}{dx} = 0$ ——— (8)

using (6) in (8), we get, $\boxed{0 = 0}$

The condition (6) gives no information about C_2.

So $V(x) = C_1$ ——— (9)

where C_1 is an arbitrary constant.

As the rod is insulated from the sides and end points as well, so heat contained in the rod is same at any time.

i.e. heat is same at $t=0$ and $t \to \infty$.

The heat stored in the rod at any time t is

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$$\int_0^a \rho A \Delta x\, c\, u(x,t) = \int_a^a \rho A c\, u(x,t)\, dx$$

$$= \rho A c \int_0^a u(x,t)\, dx.$$

Since in our problem, heat stored at $t=0$ is equal to heat stored when $t \to \infty$.

Therefore,

$$\rho c A \int_0^a \lim_{t\to 0} u(x,t)\, dx = \rho c A \int_0^a \lim_{t\to\infty} u(x,t)\, dx.$$

$$\Rightarrow \rho c A \int_0^a u(x,0)\, dx = \rho c A \int_0^a V(x)\, dx \quad \because \lim_{t\to\infty} u(x,t) = V(x)$$

$$\Rightarrow \int_0^a u(x,0)\, dx = \int_0^a V(x)\, dx$$

$$\Rightarrow \int_0^a f(x)\, dx = \int_0^a C_1\, dx \quad \because \text{by } ⑤ \text{ and } ⑨$$

$$\Rightarrow \int_0^a f(x)\, dx = C_1 x \Big|_{x=0}^{a} = C_1 a.$$

$$\Rightarrow C_1 = \frac{1}{a}\int_0^a f(x)\, dx$$

Hence the unique solution is

$$⑨ \Rightarrow V(x) = \frac{1}{a}\int_0^a f(x)\, dx.$$

Example 3: 2012,

Find the steady state solution of the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}, \quad 0 < x < a, \; t > 0.$$

$$u_x(0,t) = T_0.$$

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$$-K\frac{\partial u(a,t)}{\partial x} = h[u(a,t) - \bar{T}_1], \quad t>0$$

$$u(x,0) = f(x), \quad 0<x<a.$$

Solution:

Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}, \quad 0<x<a, \quad t>0 \quad \text{——} (1)$$

$$u(0,t) = \bar{T}_0, \quad -K\frac{\partial u(a,t)}{\partial x} = h[u(a,t) - \bar{T}_1] \quad \text{——} (2)$$

$$u(x,0) = f(x), \quad 0<x<a, \quad \text{——} (3)$$

To find steady state solution we take $\lim_{t\to\infty}$ of the given problem and $\lim_{t\to\infty} u(x,t) = v(x)$ where $v(x)$ is steady state solution.

$(1) \Rightarrow \frac{d^2v}{dx^2} = 0$.(as solved in last Prob)

$$\Rightarrow v(x) = C_1 + C_2 x \quad \text{——} (4)$$

$(2) \Rightarrow \lim_{t\to\infty} u(0,t) = \bar{T}_0 \Rightarrow v(0) = \bar{T}_0 \quad \text{——} (5)$

and $-K\lim_{t\to\infty}\frac{\partial u(a,t)}{\partial x} = \lim_{t\to\infty} h[u(a,t) - \bar{T}_1]$.

$$\Rightarrow -K\frac{dv(a)}{dx} = h[v(a) - \bar{T}_1].$$

$$\Rightarrow \frac{dv(a)}{dx} = \frac{h}{K}(\bar{T}_1 - v(a)) \quad \text{——} (6)$$

$(3) \Rightarrow \lim_{t\to\infty} u(x,0) = f(x)$

$$\Rightarrow v(x) = f(x), \quad 0<x<a \quad \text{——} (7)$$

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④ & ⑤ $\Rightarrow v(0) = C_1 \quad \Rightarrow \boxed{C_1 = \bar{T}_0}$

④ $\Rightarrow v(x) = \bar{T}_0 + C_2 x$

$\Rightarrow v(a) = \bar{T}_0 + C_2 a$ ——— ⑧

$\Rightarrow \frac{dv(a)}{dx} = C_2$

⑥ $\Rightarrow \frac{h}{K}[\bar{T}_1 - v(a)] = C_2$

using ⑧ in above relation, we get,

$\frac{h}{K}[\bar{T}_1 - \bar{T}_0 - C_2 a] = C_2$

$\Rightarrow h(\bar{T}_1 - \bar{T}_0 - C_2 a) = K C_2$

$\Rightarrow h(\bar{T}_1 - \bar{T}_0) - haC_2 = KC_2$

$\Rightarrow h(\bar{T}_1 - \bar{T}_0) = (ha + K) C_2$

$\Rightarrow C_2 = \frac{h(\bar{T}_1 - \bar{T}_0)}{ha + K}$.

Hence the steady state solution is

$$v(x) = \bar{T}_0 + \frac{h(\bar{T}_1 - \bar{T}_0)}{ha + K} x$$

Example 4:

Find the steady state and transient solutions of the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{K} \frac{\partial u}{\partial t}, \quad 0 < x < a, \ t > 0.$$

$$u(0, t) = \bar{T}_0, \quad \frac{\partial u(a, t)}{\partial x} = 0$$

$$u(x, 0) = f(x), \quad 0 < x < a,$$

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Solution.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$, $0<x<a$, $t>0$ ——— (1)

$u(0,t) = T_0$, $\frac{\partial u}{\partial x}(a,t) = 0$ ——— (2)

$u(x,0) = f(x)$, $0<x<a$, ——— (3)

First we find steady state solution of the given problem. For this we take $\lim_{t\to\infty}$ of the problem and $\lim_{t\to\infty} u(x,t) = v(x)$, where $v(x)$ is steady state solution.

(1) $v(x) = C_1 + C_2 x$ ——— (4) (same as last problem)

(2) $\Rightarrow \lim_{t\to\infty} u(0,t) = T_0 = v(0) = T_0$ ——— (5)

and $\lim_{t\to\infty} \frac{\partial u(a,t)}{\partial x} = 0 \Rightarrow \frac{dv(a)}{dx} = 0$ ——— (6)

(3) $\Rightarrow$ $v(x) = f(x)$, $0<x<a$ ——— (7)

(4) $\Rightarrow$ $v(0) = C_1 \Rightarrow \boxed{C_1 = T_0}$

So $v(x) = T_0 + C_2 x$. $\Rightarrow v(a) = T_0 + C_2 a$

$\frac{dv(a)}{dx} = C_2 \Rightarrow \boxed{C_2 = 0}$ using (6)

Thus the steady state solution is

$v(x) = T_0$

Now we find transient solution, of the given problem, which is given by

$w(x,t) = u(x,t) - v(x)$

$\Rightarrow u(x,t) = w(x,t) + v(x)$.

(1) $\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2 v(x)}{\partial x^2}^{\to 0} = \frac{1}{K}\left(\frac{\partial w(x,t)}{\partial t} + \frac{\partial v(x)}{\partial t}^{\to 0}\right)$

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$$\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{1}{K}\frac{\partial w(x,t)}{\partial t} \quad \text{——— (8)}$$

From $u(x,t) = w(x,t) + v(x)$, (2) and (3) give us

(2) $w(0,t) + v(0) = T_0$

$\Rightarrow w(0,t) + T_0 = T_0$

$\Rightarrow w(0,t) = 0$ ——— (9)

and $\frac{\partial w(a,t)}{\partial x} + \frac{\partial v(a)}{\partial x} = 0$

$\Rightarrow \frac{\partial w(a,t)}{\partial x} + 0 = 0$ ∵ using (6)

$\Rightarrow \frac{\partial w(a,t)}{\partial x} = 0$ ——— (10)

(3) $w(x,0) = g(x)$ (say) ——— (11) (Solved in example ①)

Here for $w(x,t)$, the boundary conditions are homogeneous so we can solve this problem (8) to (11) by separation of variables method.

Let $w(x,t) = X(x)T(t)$.

Then (8) gives us,

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{KT}\frac{dT}{dt} \quad \text{——— (12)}$$ see page (166)

Since L.H.S is a function of x only and R.H.S is a function of t only so both sides are equal only if they are equal to some constant (say) $-\lambda^2$ where $\lambda \in \mathbb{R}$.

Self proceed, see problem ① page (166) onward.

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Exercise

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Find the steady state solution of the problem

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) = c\rho\frac{\partial u}{\partial t}, \quad 0<x<a, \quad t>0$$

$$u(0,t) = T_0, \quad u(a,t) = T_1, \quad t>0$$

$$u(x,0) = \frac{T_1}{a}x, \quad 0<x<a.$$

Where K is given by

$$K(x) = b + d_1 x \; ; \; (b, d_1 \text{ are constants}).$$

Special

Solution.

Given that $\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right) = c\rho\frac{\partial u}{\partial t}$ ——— (1)

$; K = b + d_1 x$

$$u(0,t) = T_0, \quad u(a,t) = T_1 \quad \text{——— (2)}$$

$$u(x,0) = \frac{T_1}{a}x \quad \text{——— (3)}$$

To find steady state solution, we take $\lim_{t\to\infty}$ of the given problem and $\lim_{t\to\infty} u(x,t) = v(x)$ where $v(x)$ is steady state solution.

$$(1) \Rightarrow \frac{\partial}{\partial x}\left(K\frac{\partial v(x)}{\partial x}\right) = c\rho\cancel{\frac{\partial v(x)}{\partial t}}^{\,0}$$

$$\Rightarrow \frac{d}{dx}\left(K\frac{dv}{dx}\right) = 0$$

d_1 کونسٹنٹ ہے but not differential

$$\Rightarrow \frac{d}{dx}\left((b+d_1x)\frac{dv}{dx}\right) = 0$$

$$(b+d_1x)\frac{d^2v}{dx^2} + d_1\frac{dv}{dx} = 0$$

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Put $\frac{d\theta}{dx} = L$

Then $d_1 L + (b + d_1 x)\frac{dL}{dx} = 0$

$$\Rightarrow \frac{dL}{dx} = \frac{-d_1 L}{b + d_1 x}$$

$$\Rightarrow \frac{dL}{L} = -\frac{d_1}{b + d_1 x}\, dx$$

Integrating, we get

$$\ln L = -\ln|b + d_1 x| + \ln C_1$$

$$\ln L = -\ln|b + d_1 x| + \ln C_1$$

$$\ln L = \ln \frac{C_1}{b + d_1 x}$$

$$\Rightarrow L = \frac{C_1}{b + d_1 x}$$

By backsubsitution $L = dV/dx$

$$\frac{dv}{dx} = \frac{C_1}{b + d_1 x}$$

$$dV = \frac{C_1}{b + d_1 x}\, dx$$

Integrating, we get,

$$V(x) = \frac{C_1}{d_1} \ln|b + d_1 x| + C_2 \quad \text{——} (4)$$

(2) $\Rightarrow V(0) = T_0$, $V(a) = T_1$

(3) $\Rightarrow V(x) = \frac{T_1}{a} x$.

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④ $\Rightarrow V(0) = \frac{C_1}{d_1} \ln|b| + C_2 = \bar{T}_0$

and $V(a) = \frac{C_1}{d_1} \ln|b+ad_1| + C_2 = \bar{T}_1$

$\Rightarrow \frac{C_1}{d_1} \ln|b| + C_2 = \bar{T}_0$ ——— ⑤

and $\frac{C_1}{d_1} \ln|b+ad_1| + C_2 = \bar{T}_1$ ——— ⑥

$\Rightarrow \frac{C_1}{d_1} \ln|b| + C_2 - \bar{T}_0 = 0$

$\frac{C_1}{d_1} \ln|b+ad_1| + C_2 - \bar{T}_1 = 0$

$$\frac{C_1}{-\bar{T}_1 + \bar{T}_0} = \frac{-C_2}{-\frac{1}{d_1}\bar{T}_1 \ln|b| + \frac{1}{d_1}\bar{T}_0 \ln|b+ad_1|} = \frac{1}{\frac{1}{d_1}\ln|b| - \frac{1}{d_1}\ln|b+ad_1|}$$

$$\Rightarrow C_1 = \frac{\bar{T}_0 - \bar{T}_1}{\frac{1}{d_1}(\ln|b| - \ln|b+ad_1|)} = \frac{d_1(\bar{T}_0 - \bar{T}_1)}{\ln|b| - \ln|b+ad_1|}$$

$$\Rightarrow C_1 = \frac{d_1(\bar{T}_0 - \bar{T}_1)}{\ln|b| - \ln|b+ad_1|} = A \text{ (say)}$$

and

$$C_2 = \frac{\frac{1}{d_1}(\bar{T}_1 \ln|b| - \bar{T}_0 \ln|b+ad_1|)}{\frac{1}{d_1}(\ln|b| - \ln|b+ad_1|)}$$

$$\Rightarrow C_2 = \frac{\bar{T}_1 \ln|b| - \bar{T}_0 \ln|b+ad_1|}{\ln|b| - \ln|b+ad_1|} = B \text{ (say)}$$

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Thus the steady state solution is

$$\Rightarrow V(x) = \frac{A}{d_1} \ln|b+d_1 x| + B$$

where $A = \dfrac{d_1(\bar{T}_0 - \bar{T}_1)}{\ln|b| + \ln|b+ad_1|}$

and $B = \dfrac{\bar{T}_1 \ln|b| - \bar{T}_0 \ln|b+ad_1|}{\ln|b| + \ln|b+ad_1|}$

Question 2.

Find the steady state solution of the inhomogeneous problem

$$\frac{\partial^2 u}{\partial x^2} + s = \frac{1}{k}\frac{\partial u}{\partial t}, \quad 0<x<a, \quad t>0$$

where $s = \alpha - \beta^2 u$

$u(0,t) = \bar{T}_0$, $u(a,t) = \bar{T}_1$, $t>0$.

Solution.

Given that $\dfrac{\partial^2 u}{\partial x^2} + \alpha - \beta^2 u = \dfrac{1}{k}\dfrac{\partial u}{\partial t}$ ——— (1)

$u(0,t) = \bar{T}_0$, $u(a,t) = \bar{T}_1$ ——— (2)

To obtain steady state solution of the given problem, we take $\lim_{t\to\infty}$ of the problem and $\lim_{t\to\infty} u(x,t) = V(x)$ where $V(x)$ is steady state solution.

$$(1) \Rightarrow \frac{d^2V(x)}{dx^2} + \alpha - \beta^2 V(x) = 0$$

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$$\Rightarrow \frac{d^2V(x)}{dx^2} - \beta^2 V(x) = -\alpha$$

The characteristic equation is

$$D^2 - \beta^2 = 0$$

$$\Rightarrow D^2 = \beta^2 \quad \Rightarrow D = \pm\beta \qquad \text{(real, distinct)}$$

Thus $V_c(x) = C_1 e^{\beta x} + C_2 e^{-\beta x}$ ——— (3)

For P.I. $= \dfrac{-\alpha}{D^2 - \beta^2} = \dfrac{-\alpha}{-\beta^2\left(1 - \frac{D^2}{\beta^2}\right)}$

$$= \left(1 - \frac{D^2}{\beta^2}\right)^{-1} (\alpha/\beta^2) = \frac{\alpha}{\beta^2} = V_p(x).$$

The general solution is

$$V(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + \frac{\alpha}{\beta^2} \qquad \text{——— (4)}$$

(2) $\Rightarrow V(0) = \bar{T}_0$, $V(a) = \bar{T}_1$

(4) $\Rightarrow V(0) = C_1 + C_2 + \alpha/\beta^2$; $V(a) = C_1 e^{\beta a} + C_2 e^{-\beta a} + \frac{\alpha}{\beta^2}$

$$\Rightarrow \quad C_1 + C_2 + \alpha/\beta^2 = \bar{T}_0$$
$$C_1 e^{\beta a} + C_2 e^{-\beta a} + \frac{\alpha}{\beta^2} = \bar{T}_1$$

$$\Rightarrow \quad C_1 + C_2 + (\alpha/\beta^2 - \bar{T}_0) = 0$$
$$C_1 e^{\beta a} + C_2 e^{-\beta a} + (\alpha/\beta^2 - \bar{T}_1) = 0$$

$$\frac{C_1}{(\frac{\alpha}{\beta^2} - \bar{T}_1) - e^{-\beta a}(\frac{\alpha}{\beta^2} - \bar{T}_0)} = \frac{-C_2}{(\frac{\alpha}{\beta^2} - \bar{T}_1) - e^{\beta a}(\frac{\alpha}{\beta^2} - \bar{T}_0)} = \frac{1}{e^{-\beta a} - e^{\beta a}}$$

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$$\Rightarrow C_1 = \frac{(\alpha/\beta^2 - \bar{T}_1) - e^{-\beta a}(\alpha/\beta^2 - \bar{T}_0)}{-2\sinh(\beta a)} = A \text{ (say)}$$

and

$$C_2 = \frac{e^{\beta a}(\frac{\alpha}{\beta^2} - \bar{T}_0) - (\frac{\alpha}{\beta^2} - \bar{T}_1)}{-2\sinh(\beta a)} = B \text{ (say)}$$

Thus the steady state solution of the given problem is

$$\Rightarrow V(x) = Ae^{\beta x} + Be^{-\beta x} + \alpha/\beta^2$$

where

$$A = \frac{e^{-\beta a}(\alpha/\beta^2 - \bar{T}_0) - (\alpha/\beta^2 - \bar{T}_1)}{2\sinh(\beta a)}$$

$$B = \frac{(\alpha/\beta^2 - \bar{T}_1) - e^{\beta a}(\alpha/\beta^2 - \bar{T}_0)}{2\sinh(\beta a)}.$$

Question 3.

Solve the inhomogeneous problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \frac{T}{a^2}, \quad 0 < x < a, \quad t > 0$$

$$u(0,t) = \bar{T}_0, \quad \frac{\partial u(a,t)}{\partial x} = 0, \quad t > 0$$

$$u(x,0) = \bar{T}_0, \quad 0 < x < a$$

Solution.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} + \frac{T}{a^2}$ ——— (1)

$$u(0,t) = \bar{T}_0, \quad \frac{\partial u(a,t)}{\partial x} = 0 \quad \text{——— (2)}$$

$$u(x,0) = T_0 \quad \text{——— (3)}$$

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First we obtain steady state solution of the given problem. Take limits of the problem and $\lim_{t\to\infty} u(x,t) = v(x)$, where $v(x)$ is steady state solution of the problem.

① $\Rightarrow \frac{d^2v}{dx^2} = \frac{1}{k}\frac{\partial v}{\partial t}^{0} + \frac{T}{a^2}$

$\Rightarrow \frac{d^2v}{dx^2} = \frac{T}{a^2}$

Integrating, we get

$\frac{dv}{dx} = \frac{T}{a^2}x + C_1$ — (*)

$v(x) = \frac{T}{a^2}\cdot\frac{x^2}{2} + C_1x + C_2$ — (4)

② $\Rightarrow v(0) = T_0$, $\frac{dv(a)}{dx} = 0$ — (5)

(4) $v(0) = C_2 \Rightarrow \boxed{C_2 = T_0}$

(*) $\Rightarrow \frac{dv(a)}{dx} = \frac{T}{a^2}a + C_1 = \frac{T}{a} + C_1$

(5) $\Rightarrow \frac{T_0}{a} + C_1 = 0 \Rightarrow \boxed{C_1 = -\frac{T_0}{a}}$

Thus the steady state solution is

(4) $\Rightarrow v(x) = \frac{T}{a^2}\frac{x^2}{2} - \frac{T_0}{a}x + T_0$

$= \frac{Tx^2 - 2ax}{2a^2} + T_0$

Or $v(x) = T_0 + \frac{Tx(x-2a)}{2a^2}$

Now we find transient solution of the problem.

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i.e. $w(x,t) = u(x,t) - v(x)$

$\Rightarrow u(x,t) = w(x,t) + v(x)$.

① $\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} + \frac{\partial^2 v(x)}{\partial x^2} = \frac{1}{K}\frac{\partial w(x,t)}{\partial t} + \frac{\bar{I}}{a^2}$

$\Rightarrow \frac{d^2 w(x,t)}{dx^2} + \frac{d^2 v(x)}{dx^2} = \frac{1}{K}\frac{dw(x,t)}{dt} + \frac{\bar{I}}{a^2}$

$\Rightarrow \frac{d^2 w(x,t)}{dx^2} + \frac{\bar{I}}{a^2} = \frac{1}{K}\frac{dw(x,t)}{dt} + \frac{\bar{I}}{a^2}$

$\Rightarrow \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{1}{K}\frac{\partial w(x,t)}{\partial t}$ ⟶ ⑥

From $u(x,t) = w(x,t) + v(x)$ ② and ③ yeid, $w(0,t) + v(0) = \bar{I}_0$

$\Rightarrow w(0,t) + \bar{I}_0 = \bar{I}_0 \Rightarrow w(0,t) = 0$ ⟶ ⑦

$\frac{\partial w(a,t)}{\partial x} + \frac{\partial v(a)}{\partial x} = 0$

$\frac{\partial w(a,t)}{\partial x} + \frac{\bar{I}}{a} = 0$ ⟶ ⑧ ∵ using ⑤

③ $w(x,0) + v(x) = \bar{I}_0$

$\Rightarrow w(x,0) = \bar{I}_0 - v(x)$

$\Rightarrow w(x,0) = \bar{I}_0 - \frac{\bar{I}}{a^2}\cdot\frac{x^2}{2} + \frac{\bar{I}_0}{a}x - \bar{I}_0$

$\Rightarrow w(x,0) = \frac{\bar{I}_0}{a}x - \frac{\bar{I}x^2}{2a^2} = \frac{2\bar{I}ax - \bar{I}x^2}{2a^2}$

$w(x,0) = \frac{\bar{I}x(2a-x)}{2a^2}$ ⟶ ⑨

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Here for $w(x,t)$ the boundary conditions are homogeneous so we can solve this problem ⑥ to ⑨ by separation of variables method.

Let $w(x,t) = X(x)T(t)$.

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کیونکہ T آگے اس question میں موجود ہے

Then ⑥ gives us

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{T}{KT'}\frac{dT}{dt}$$ ⑩ see page (166)

Proceed as example 1 (page 166 examp1)

Question 4.

Find the steady state solution of the problem

$$\frac{\partial^2 u}{\partial x^2} - \gamma^2(u-T) = \frac{1}{K}\frac{\partial u}{\partial t}, \quad 0<x<a,\ t>0 \quad \text{—— ①}$$

$$u(0,t) = T_0, \quad u(a,t) = T_1, \quad t>0 \quad \text{—— ②}$$

$$u(x,0) = 0, \quad 0<x<a. \quad \text{—— ③}$$

Solution.

To find steady state solution of the given problem we take $\lim_{t\to\infty}$ of the problem and $\lim_{t\to\infty} u(x,t) = v(x)$, where $v(x)$ is steady state solution.

$$① \Rightarrow \frac{d^2v(x)}{dx^2} - \gamma^2(v(x) - T) = 0$$

$$\Rightarrow \frac{d^2v(x)}{dx^2} - \gamma^2 v(x) + \gamma^2 T = 0$$

$$\Rightarrow D^2v(x) - \gamma^2 v(x) = -\gamma^2 T \qquad \because D^2 = \frac{d^2}{dx^2}$$

The Characteristic equation is

$$D^2 - \gamma^2 = 0$$

$$D = \pm\gamma$$

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So $V_c(x) = C_1 e^{\gamma x} + C_2 e^{-\gamma x}$

For P.I. $= \dfrac{-\gamma^2 \bar{T}}{D^2 - \gamma^2} = \dfrac{-\gamma^2 \bar{T}}{-\gamma^2\left(1 - \frac{D^2}{\gamma^2}\right)}$

$= \dfrac{\bar{T}}{\left(1 - \frac{D^2}{\gamma^2}\right)} = \left(1 - \frac{D^2}{\gamma^2}\right)^{-1}(\bar{T}) = \bar{T} = V_p(x)$

Thus $V(x) = C_1 e^{\gamma x} + C_2 e^{-\gamma x} + \bar{T}$ ——— (4)

(2) $V(0) = \bar{T}_0$, $V(a) = \bar{T}_1$ ——— (5)

(4) $\Rightarrow V(0) = C_1 + C_2 + \bar{T}$

$V(a) = C_1 e^{\gamma a} + C_2 e^{-\gamma a} + \bar{T}$

(5) $\Rightarrow$ $C_1 + C_2 + \bar{T} = \bar{T}_0$

$C_1 e^{\gamma a} + C_2 e^{-\gamma a} + \bar{T} = \bar{T}_1$

$\Rightarrow$ $C_1 + C_2 + (\bar{T} - \bar{T}_0) = 0$

$e^{\gamma a} C_1 + e^{-\gamma a} C_2 + (\bar{T} - \bar{T}_1) = 0$

$\Rightarrow \dfrac{C_1}{(\bar{T} - \bar{T}_1) - e^{-\gamma a}(\bar{T} - \bar{T}_0)} = \dfrac{-C_2}{(\bar{T} - \bar{T}_1) - e^{\gamma a}(\bar{T} - \bar{T}_0)} = \dfrac{1}{e^{-\gamma a} - e^{\gamma a}}$

$\Rightarrow C_1 = \dfrac{(\bar{T} - \bar{T}_1) - e^{-\gamma a}(\bar{T} - \bar{T}_0)}{-2\sinh(\gamma a)} = A$ (say)

and

$C_2 = \dfrac{e^{\gamma a}(\bar{T} - \bar{T}_0) - (\bar{T} - \bar{T}_1)}{-2\sinh(\gamma a)} = B$ (say).

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Thus the steady state solution is

$\textcircled{1} \Rightarrow V(x) = Ae^{\gamma x} + Be^{-\gamma x} + \bar{T}$

where $A = \dfrac{e^{-\gamma a}(\bar{T}-\bar{T}_0) - (\bar{T}-\bar{T}_1)}{2\sinh(\gamma a)}$

and $B = \dfrac{(\bar{T}-\bar{T}_1) - e^{\gamma a}(\bar{T}-\bar{T}_0)}{2\sinh(\gamma a)}$

Question 5.

Find a formal solution of the heat flow problem described by

$$\frac{\partial u}{\partial t} = \frac{1}{k}\frac{\partial^2 u}{\partial x^2} + P(x), \quad 0<x<a, \ t>0.$$

$$u(0,t) = u_1, \ u(a,t) = u_2, \quad t>0.$$

$$u(x,0) = f(x), \quad 0<x<a.$$

Solution.

Given that $\dfrac{\partial u}{\partial t} = \dfrac{1}{k}\dfrac{\partial^2 u}{\partial x^2} + P$

$$\Rightarrow \frac{\partial u}{\partial t} - P = \frac{1}{k}\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = k\frac{\partial u}{\partial t} - Pk \quad \text{——— (1)}$$

$$u(0,t) = u_1, \ u(a,t) = u_2 \quad \text{——— (2)}$$

$$u(x,0) = f(x) \quad \text{——— (3)}$$

First we find steady state solution of the given problem we take $\lim_{t\to\infty}$ of the problem and $\lim_{t\to\infty} u(x,t) = u(x)$, where $u(x)$ is steady state solution.

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① $\Rightarrow \frac{\partial^2 v(x)}{\partial x^2} = \cancel{K \frac{\partial v(x)}{\partial t}}^{0} - PK$

$\Rightarrow \frac{d^2 v(x)}{dx^2} = -PK$

$\Rightarrow \frac{d}{dx}\left(\frac{dv}{dx}\right) = -PK$

Integrating, we get

$$\frac{dv(x)}{dx} = -PKx + G_1 \quad \text{——} \quad (4)$$

Again integrating, we have

$$v(x) = -PK\frac{x^2}{2} + G_1 x + C_2 \quad \text{——} \quad (5)$$

② $\Rightarrow v(0) = u_1, \quad v(a) = u_2$ —— (6)

③ $\Rightarrow v(x) = f(x)$ —— (7)

⑤ $\Rightarrow v(0) = C_2$

$$v(a) = -PK\frac{a^2}{2} + G_1 a + C_2$$

⑥ $\Rightarrow \boxed{C_2 = u_1}$ and $u_2 = -PK\frac{a^2}{2} + G_1 a + u_1$

$\Rightarrow u_2 - u_1 + PK a^2/2 = G_1 a$

$\Rightarrow G_1 = \frac{1}{a}\left(u_2 - u_1 + PK\frac{a^2}{2}\right)$.

Thus the steady state solution is

⑧ $\Rightarrow v(x) = -PK\frac{x^2}{2} + \left(u_2 - u_1 + PK\frac{a^2}{2}\right)\frac{x}{a} + u_1$

Now we find transient solution which is given by

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$$W(x,t) = u(x,t) - v(x)$$

$$\Rightarrow u(x,t) = W(x,t) + v(x)$$

$$(1) \Rightarrow \frac{\partial^2 W(x,t)}{\partial x^2} + \frac{\partial^2 v(x)}{\partial x^2} = \frac{K}{} \frac{\partial W(x,t)}{\partial t} - PK$$

$$\Rightarrow \frac{\partial^2 W(x,t)}{\partial x^2} - PK = K \frac{\partial W(x,t)}{\partial t} - PK$$

$$\Rightarrow \frac{\partial^2 W(x,t)}{\partial x^2} = K \frac{\partial W(x,t)}{\partial t} \quad \text{——— (8)}$$

$$(2) \Rightarrow W(0,t) + v(0) = u_1$$

$$\Rightarrow W(0,t) + u_1 = u_1 \Rightarrow W(0,t) = 0 \quad \text{——— (9)}$$

and $W(a,t) + v(a) = u_2$

$$\Rightarrow W(a,t) + u_2 = u_2 \Rightarrow W(a,t) = 0 \quad \text{——— (10)}$$

$$(3) \Rightarrow W(x,0) + v(x) = f(x)$$

$$\Rightarrow W(x,t) = f(x) - v(x) = g(x). \quad \text{——— (11)}$$

Proceed as example 1: (see page (166)).

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One Dimensional Wave Equation

Special

Derive one dimensional wave equation. 2014 & 2018 1A

To derive one dimensional wave equation, we consider the transverse vibrations of a stretched string of finite length. To obtain the equation in standard form we make the following assumptions.

① The string is stretched in a tight position and fixed at the end points.

② The tension in the string is constant and is along the tangent to the curve of the string at any time.

③ The weight of the string is small compared to the tension in it.

(4) The deflection (displacement) of the string is small compared to the length of the string.

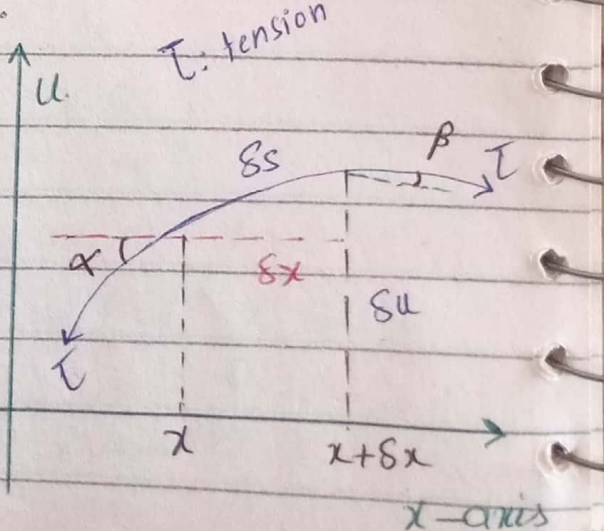
(5) The slope of the displaced string is small compared to 1.

(6) There are only pure transverse vibrations.

(7) The density of the string is constant.

(8) There is no external force.

We suppose that the string is lying along x-axis so that its displacement at any point x and at any time t may be denoted by u(x, t).



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Let us consider an element of the vibrating string of length δs whose magnified form is shown in the fig.

Since this element of the string is moving in the direction $\perp$ to the x-axis.

Therefore the resultant force on the element must be in the direction of motion.

The total force in the direction of motion is given by

negative sign due to opposite direc

$$F = T\sin\beta + (-T\sin\alpha)$$
$$= T\sin\beta - T\sin\alpha$$

The Newton's second law of motion is

$$F = m\vec{a}$$

$$\Rightarrow T\sin\beta - T\sin\alpha = \rho\,\delta s \cdot \frac{\partial^2 u(x,t)}{\partial t^2} \qquad \because \rho = \frac{\text{mass}}{\text{length}} = \frac{m}{\delta s}$$

$$\Rightarrow \sin\beta - \sin\alpha = \frac{\rho}{T}\,\delta s \cdot \frac{\partial^2 u(x,t)}{\partial t^2} \qquad \text{---------- (1)}$$

Now $$\delta s = \sqrt{(\delta x)^2 + (\delta u)^2}$$
$$= \delta x \sqrt{1 + \left(\frac{\delta u}{\delta x}\right)^2}$$
$$= \delta x \sqrt{1 + u_x^2} \qquad \because u_x = \frac{\partial u}{\partial x}$$

$$\Rightarrow \delta s \simeq \delta x \qquad \text{---------- (2)} \because u_x \text{ is very small}$$

see point(5) slope is small

By the same reason we know that α and β are very small.

So $\sin\alpha \simeq \tan\alpha$

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Slope of string at $x = \frac{\partial}{\partial x} u(x,t) = \tan\alpha$

$\Rightarrow \sin\alpha = \frac{\partial}{\partial x} u(x,t)$ ——— (3)

Slope of string at $x+\delta x = \frac{\partial}{\partial x} u(x+\delta x, t) = \tan\beta$

$\Rightarrow \sin\beta = \frac{\partial}{\partial x} u(x+\delta x, t)$ ——— (4)

Subsituting (2), (3) and (4) in (1), we get,

$$\frac{\partial u(x+\delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x} = \frac{\rho}{T}\delta x \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\Rightarrow \frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x} = \frac{\rho}{T}\frac{\partial^2 u(x,t)}{\partial t^2}$$

Taking $\lim_{\delta x \to 0}$, we get,

$$\lim_{\delta x\to 0} \frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x} = \lim_{\delta x\to 0} \frac{\rho}{T}\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$\Rightarrow \frac{\partial u_x}{\partial x} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Where $\frac{\rho}{T} = \frac{1}{c^2}$ which is required one dimensional wave equation.

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Special

Example 1: 2012,

A string of length a is stretched between two fixed points and is made to vibrate by plucking it at the middle point through a short distance h. Formulate and solve the problem.

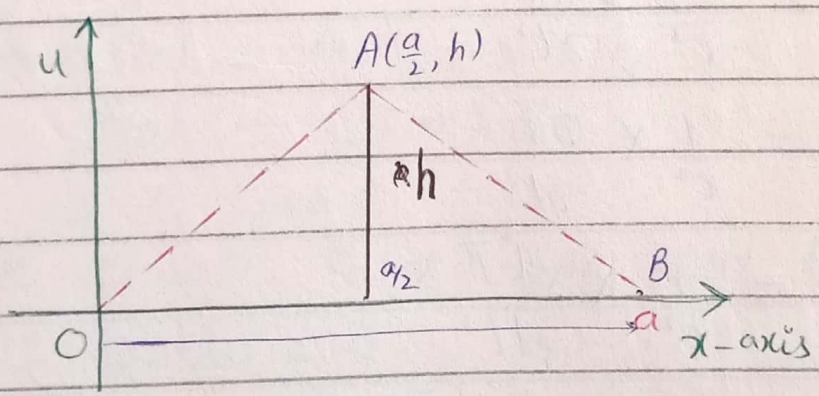
Solution.

Let the string be fixed at the points $x=0$ and $x=a$. Its vibrations are described by the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{---------- (1)}$$

The boundary conditions are given by

$$u(0,t) = 0 \;,\; u(a,t) = 0 \quad \text{---------- (2)}$$



Equation of line $\overrightarrow{OA}$: $u-0 = \frac{h-0}{\frac{a}{2}-0}(x-0)$

$$\Rightarrow u = \frac{2h}{a}x$$

Equation of line $\overrightarrow{AB}$: $u-0 = \frac{h-0}{\frac{a}{2}-a}(x-a)$

$$\Rightarrow u = -\frac{2h}{a}(x-a)$$

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The initial conditions are given by

$$u(x,0) = \begin{cases} \frac{2h}{a}x & \text{if } 0 \le x \le a/2 \\ -\frac{2h}{a}(x-a) & \text{if } \frac{a}{2} < x \le a \end{cases} \longrightarrow (3)$$

$$\frac{\partial u}{\partial t}(x,0) = 0$$

The problem has been formulated in the form of equation (1), (2) and (3).

Now we solve the problem by using separation of variables method.

Let $u(x,t) = X(x)\,T(t)$ ——— (4)

Then (1) becomes

$$\frac{\partial^2 XT}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 XT}{\partial t^2}$$

$$\Rightarrow T\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2}X\frac{\partial^2 T}{\partial t^2}$$

$$\Rightarrow T\frac{d^2 X}{dx^2} = \frac{1}{c^2}X\frac{d^2 T}{dt^2}$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Tc^2}\frac{d^2 T}{dt^2} \quad \text{——— (5)}$$

Since L.H.S is a function of x only and R.H.S is a function of t only, both sides are equal if they are equal to some constant, say, $-\lambda^2$ where $\lambda \in \mathbb{R} \setminus \{0\}$.

(5) $\Rightarrow$ $\frac{1}{X}\frac{d^2 X}{dx^2} = -\lambda^2$

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and $\frac{1}{Tc^2}\frac{d^2T}{dt} = -\lambda^2$

$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$ ——— (6)

and $\frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$ ——— (7)

(6) $\Rightarrow (D^2 + \lambda^2)X = 0$ $\quad \because D^2 = \frac{d^2}{dx^2}$

The characteristic eq. is

$D^2 + \lambda^2 = 0$

$\Rightarrow D = \pm i\lambda$

The general solution is

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ ——— (8)

(2) $\Rightarrow X(0)T(t) = 0$; $X(a)T(t) = 0$

$\Rightarrow X(0) = 0$; $X(a) = 0$ $\quad \because T(t) \neq 0$.

(8) $\Rightarrow X(0) = C_1 \Rightarrow \boxed{C_1 = 0}$

and $X(a) = C_2 \sin\lambda a$

$\Rightarrow C_2 \sin\lambda a = 0$

we choose $C_2 \neq 0$ for non trivial solution

$\Rightarrow \sin\lambda a = 0$

$\Rightarrow \lambda a = n\pi$; $n = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \lambda_n = \frac{n\pi}{a}$; $n = 1, 2, 3, \ldots$

which are eigenvalues and eigenfunctions are $X_n(x) = C_n \sin\left(\frac{n\pi}{a}\right)x$; $n = 1, 2, 3, \ldots$

(7) $\Rightarrow (\Delta^2 + \lambda^2 c^2)T = 0$ $\quad \because \Delta^2 = d^2/dt^2$

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The auxiliary equation is

$$\Delta^2 + \lambda^2 c^2 = 0$$

$$\Rightarrow \Delta = \pm i\lambda c$$

The general solution for T is

$$T(t) = C_3 \cos(\lambda c t) + C_4 \sin(\lambda c t)$$

or $$T_n(t) = C_n' \cos \lambda_n c t + C_n^* \sin \lambda_n c t$$

$$\Rightarrow T_n(t) = C_n' \cos\left(\frac{n\pi c}{a}\right)t + C_n^* \sin\left(\frac{n\pi c}{a}\right)t \quad , n = 1, 2, 3 \cdots$$

(4) $\Rightarrow$

$$u_n(x,t) = X_n(x)\, T_n(t)$$

$$= C_n \sin\left(\frac{n\pi}{a}\right)x \left\{ C_n' \cos\left(\frac{n\pi c}{a}\right)t + C_n^* \sin\left(\frac{n\pi c}{a}\right)t \right\}$$

$$= \sin\left(\frac{n\pi}{a}\right)x \left\{ a_n' \cos\left(\frac{n\pi c}{a}\right)t + b_n' \sin\left(\frac{n\pi c}{a}\right)t \right\}.$$

where $a_n' = C_n C_n'$, $b_n' = C_n C_n^*$

By the principle of superposition, the linear combination of the above solution is also a solution.

i.e.

$$u(x,t) = \sum_n d_n u_n(x,t) = \sum_n d_n \sin\left(\frac{n\pi}{a}\right)x \left\{ a_n' \cos\left(\frac{n\pi c}{a}\right)t + b_n' \sin\left(\frac{n\pi c}{a}\right)t \right\}$$

$$\Rightarrow u(x,t) = \sum_n \sin\left(\frac{n\pi}{a}\right)x \left\{ a_n \cos\left(\frac{n\pi c}{a}\right)t + b_n \sin\left(\frac{n\pi c}{a}\right)t \right\} \longrightarrow (9)$$

where $a_n = d_n a_n'$, $b_n = d_n b_n'$.

$$\Rightarrow u(x,0) = \sum_n a_n \sin\left(\frac{n\pi}{a}\right)x \longrightarrow (10)$$

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③ $\Rightarrow \sum_n a_n \sin\left(\frac{n\pi}{a}\right)x = \begin{cases} \frac{2h}{a}x & \text{if } 0 \le x \le a/2 \\ -\frac{2h}{a}(x-a) & \text{if } \frac{a}{2} < x \le a \end{cases}$

and $u_t(x,0) = 0 = g(x)$ (say).

④ $\Rightarrow u_t(x,0) = \sum_n \sin\left(\frac{n\pi}{a}\right)x \left\{ a_n \left(\frac{n\pi c}{a}\right) \sin\left(\frac{n\pi c}{a}\right)t + b_n \left(\frac{n\pi c}{a}\right) \cos\left(\frac{n\pi c}{a}\right)t \right\}\Big|_{t=0}$

$\Rightarrow u_t(x,0) = \sum_n \sin\left(\frac{n\pi}{a}\right)x \left\{ b_n \left(\frac{n\pi c}{a}\right) \right\}.$

$\Rightarrow g(x) = \sum_n b_n \left(\frac{n\pi c}{a}\right) \sin\left(\frac{n\pi}{a}\right)x.$

$\Rightarrow b_n \left(\frac{n\pi c}{a}\right) = \frac{2}{a}\int_0^a g(x) \sin\left(\frac{n\pi}{a}\right)x$

$\Rightarrow b_n = 0 \quad \because g(x) = 0$

⑩ $\Rightarrow u(x,0) = \sum_n a_n \sin\left(\frac{n\pi}{a}\right)x = f(x)$ (say)

$\Rightarrow a_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}\right)x.$

$\Rightarrow a_n = \frac{2}{a}\left\{ \int_0^{a/2} \frac{2h}{a}x \sin\left(\frac{n\pi}{a}\right)x + \int_{a/2}^{a} \frac{-2h}{a}(x-a)\sin\left(\frac{n\pi}{a}\right)x\,dx \right\}$

$\Rightarrow a_n = \frac{2}{a}\left\{ \int_0^{a/2} \frac{2h}{a}x \sin\left(\frac{n\pi}{a}\right)x\,dx - \frac{2h}{a}\int_{a/2}^{a} (x-a)\sin\left(\frac{n\pi}{a}\right)x\,dx \right\}$

$\Rightarrow a_n = \frac{2h}{a^2}\left\{ I_1 - I_2 \right\}$ ——— (11)

where $I_1 = \int_0^{a/2} x \sin\left(\frac{n\pi}{a}\right)x\,dx,$

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and $I_2 = \int_{a/2}^{a} (x-a)\sin\left(\frac{n\pi}{a}\right)x\,dx$.

Now $I_1 = -\frac{a}{n\pi}x\cos\left(\frac{n\pi}{a}\right)x\Big|_0^{a/2} + \frac{a}{n\pi}\int_0^{a/2}\cos\left(\frac{n\pi}{a}\right)x\,dx$

$= -\frac{a}{n\pi}x\cos\left(\frac{n\pi}{a}\right)x\Big|_0^{a/2} + \frac{a}{n\pi}\cdot\frac{a}{n\pi}\sin\left(\frac{n\pi}{a}\right)x\Big|_0^{a/2}$

$= +\frac{a}{n\pi}\left[-x\cos\left(\frac{n\pi}{a}\right)x + \frac{a}{n\pi}\sin\left(\frac{n\pi}{a}\right)x\right]_0^{a/2}$

$= \frac{a}{n\pi}\left[-\frac{a}{2}\cos\left(n\frac{\pi}{2}\right) + 0 + \frac{a}{n\pi}\sin\left(n\frac{\pi}{2}\right) - 0\right]$

$\Rightarrow I_1 = -\frac{a^2}{2n\pi}\cos\left(n\frac{\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)$ ——— (12)

and $I_2 = -(x-a)\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}\right)x\Big|_{a/2}^{a} + \int_{a/2}^{a}\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}\right)x\,dx$

$= -(x-a)\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}\right)x\Big|_{a/2}^{a} + \frac{a^2}{n\pi}\sin\left(\frac{n\pi}{a}\right)x\Big|_{a/2}^{a}$

$= \frac{a}{n\pi}\left[-(x-a)\cos\left(\frac{n\pi}{a}\right)x + \frac{a}{n\pi}\sin\left(\frac{n\pi}{a}\right)x\right]_{a/2}^{a}$

$= \frac{a}{n\pi}\left[0 + \left(-\frac{a}{2}\right)\cos\left(n\frac{\pi}{2}\right) + \frac{a}{n\pi}\sin(n\pi) - \frac{a}{n\pi}\sin\left(n\frac{\pi}{2}\right)\right]$

$= -\frac{a^2}{2n\pi}\cos\left(n\frac{\pi}{2}\right) - \frac{a^2}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)$.

——— (13)

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Substituting (12) and (13) in (11), we get,

$$a_n = \frac{4h}{a^2}\left[\frac{-a^2}{2n\pi}\cos\left(n\frac{\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right) + \frac{a^2}{2n\pi}\cos\left(n\frac{\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)\right]$$

$$a_n = \frac{4h}{a^2}\left[2\frac{a^2}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)\right]$$

$$\Rightarrow a_n = \frac{8h}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)$$

Thus equation (9) becomes.

$$u(x,t) = \sum_n \sin\left(\frac{n\pi}{a}\right)x\left\{\frac{8h}{n^2\pi^2}\sin\left(n\frac{\pi}{2}\right)\cos\left(\frac{n\pi c}{a}\right)t + 0\right\}$$

$$= \sum_n \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{a}\right)x\sin\left(n\frac{\pi}{2}\right)\cos\left(\frac{n\pi c}{a}\right)t.$$

which is our final solution.

Example 2. 2014, New

Solve the vibrating problem when the initial and boundary conditions are given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < a, \quad t > 0.$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$

$$u(0,t) = 0, \quad u(a,t) = 0.$$

Solution.

Given that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < a \quad \text{——} (1)$$

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(B.Cs) $u(0,t) = 0$, $u(a,t) = 0$ ——— (2)

(I.Cs) $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ ——— (3)

We solve this problem by the method of separation of variables.

For this let $u(x,t) = X(x)T(t)$ ——— (4)

Then (1) $\Rightarrow \frac{\partial^2 XT}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 XT}{\partial t^2}$

$\Rightarrow T\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 T}{\partial x^2}^{0} = \frac{1}{c^2}X\frac{\partial^2 T}{\partial t^2} + \frac{1}{c^2}T\frac{\partial^2 X}{\partial t^2}^{0}$

$\Rightarrow T\frac{d^2X}{dx^2} = \frac{1}{c^2}X\frac{d^2T}{dt^2}$

$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Tc^2}\frac{d^2T}{dt^2}$ ——— (5)

Since the L.H.S of (5) is a function of x only and R.H.S is function of t only, therefore, both sides are equal if they are equal to some constant say $-\lambda^2$.

(5) $\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$, $\frac{1}{Tc^2}\frac{d^2T}{dt^2} = -\lambda^2$

$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$ ——— (6)

and $\frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$ ——— (7)

Now (6) $\Rightarrow D^2X + \lambda^2 X = 0$ $\because D^2 = \frac{d^2}{dx^2}$

$\Rightarrow (D^2 + \lambda^2)X = 0$

The char. equ is

$D^2 + \lambda^2 = 0$.

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$\Rightarrow D = \pm i\lambda$

$\Rightarrow X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ ——— (8)

$u(x,t) = X(x)T(t)$ in (2) implies that

$u(0,t) = X(0)T(t) = 0$

$\Rightarrow X(0) = 0 \quad \because T(t) \neq 0.$

and $u(a,t) = X(a)T(t) = 0$

$\Rightarrow X(a) = 0 \quad \because T(t) \neq 0.$

(8) $\Rightarrow X(0) = C_1 = 0 \Rightarrow \boxed{C_1 = 0}$

$X(a) = C_1 \cos\lambda a + C_2 \sin\lambda a = 0$

$\Rightarrow C_2 \sin\lambda a = 0 \quad \because C_1 = 0.$

For non trivial solution we choose $C_2 \neq 0$.

$\Rightarrow \sin\lambda a = 0.$

$\Rightarrow \lambda a = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \cdots$

which are eigenvalues of given problem.

(8) $\Rightarrow X_n(x) = C_n \sin\lambda_n x.$

$\Rightarrow X_n(x) = C_n \sin\left(\frac{n\pi}{a}\right)x; \quad n = 1, 2, 3, \cdots$

(7) $\Rightarrow \Delta^2 T + \lambda^2 c^2 T = 0 \quad \because \Delta^2 = d^2/dt^2$

$\Rightarrow (\Delta^2 + \lambda^2 c^2) T = 0.$

The charac. eq is

$\Delta^2 + \lambda^2 c^2 = 0$

$\Rightarrow \Delta = \pm i\lambda c.$

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$$\Rightarrow T(t) = C_3 \cos(\lambda c)t + C_4 \sin(\lambda c)t$$

or $T_n(t) = C_n' \cos(\lambda_n c)t + C_n^* \sin(\lambda_n c)t$

$$\Rightarrow T_n(t) = C_n' \cos\left(\frac{n\pi c}{a}\right)t + C_n^* \sin\left(\frac{n\pi c}{a}\right)t$$

④ $\Rightarrow u_n(x,t) = X_n(x)\, T_n(t)$

$$= C_n \sin\left(\frac{n\pi}{a}\right)x \left\{C_n' \cos\left(\frac{n\pi c}{a}\right)t + C_n^* \sin\left(\frac{n\pi c}{a}\right)t\right\}$$

$$u_n(x,t) = \sin\left(\frac{n\pi}{a}\right)x \left\{a_n' \cos\left(\frac{n\pi c}{a}\right)t + b_n' \sin\left(\frac{n\pi c}{a}\right)t\right\}$$

where $a_n' = C_n C_n'$, $b_n' = C_n C_n^*$

By the principle of superposition, the linear combination of the above solution is also a solution.

i.e.

$$u(x,t) = \sum_n d_n u_n(x,t) = \sum_n d_n \sin\left(\frac{n\pi}{a}\right)x \left\{a_n' \cos\left(\frac{n\pi c}{a}\right)t + b_n' \sin\left(\frac{n\pi c}{a}\right)t\right\}.$$

$$\Rightarrow u(x,t) = \sum_n \sin\left(\frac{n\pi}{a}\right)x \left\{a_n \cos\left(\frac{n\pi c}{a}\right)t + b_n \sin\left(\frac{n\pi c}{a}\right)t\right\} \longrightarrow ⑨$$

where $a_n = d_n a_n'$, $b_n = d_n b_n'$

The initial condition that is equation ③ gives us

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

⑨ $\Rightarrow u(x,0) = \sum_n a_n \sin\left(\frac{n\pi}{a}\right)x = f(x)$

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$$\Rightarrow a_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}\right)x\,dx. \quad \text{——— (10)}$$

and $u_t(x,0) = \sum_n \sin\left(\frac{n\pi}{a}\right)x\left\{-a_n\left(\frac{n\pi c}{a}\right)\sin\left(\frac{n\pi c}{a}\right)t + b_n\left(\frac{n\pi c}{a}\right)\cos\left(\frac{n\pi c}{a}\right)t\right\}\Big|_{t=0}$

$$= \sum_n b_n \overset{\left(\frac{n\pi c}{a}\right)}{\uparrow} \sin\left(\frac{n\pi}{a}\right)x = g(x).$$

$$\Rightarrow \left(\frac{n\pi c}{a}\right)b_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}\right)x\,dx \quad \text{——— (11)}$$

Thus final solution is

$$\text{(9)} \Rightarrow u(x,t) = \sum_n \sin\left(\frac{n\pi}{a}\right)x\left\{a_n\cos\left(\frac{n\pi c}{a}\right)t + b_n\sin\left(\frac{n\pi c}{a}\right)t\right\}.$$

where $a_n = \frac{2}{a}\int_0^a f(x)\sin\left(n\frac{\pi}{a}\right)x\,dx.$

and $\left(\frac{n\pi c}{a}\right)b_n = \frac{2}{a}\int_0^a g(x)\sin\left(n\frac{\pi}{a}\right)x\,dx.$

$$\Rightarrow b_n = \frac{2}{n\pi c}\int_0^a g(x)\sin\left(n\frac{\pi}{a}\right)x\,dx. \quad \text{——— (11)}$$

Example 3:

Solve the string problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0<x<a, \ t>0. \quad \text{——— (1)}$$

$$u(x,0) = 0, \quad u_t(x,0) = 1 \quad \text{——— (2)}$$

$$u(0,t) = 0, \quad u(a,t) = 0. \quad \text{——— (3)}$$

Solution.

See example 2 page 198. write eq (9)

The initial conditions in (2) implies that

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$u(x,0) = 0 = f(x)$ (say)

and $u_t(x,0) = 1 = g(x)$ (say)

(10) $\Rightarrow a_n = 0$

(11) $\Rightarrow b_n = \frac{2}{n\pi c}\int_0^a \sin\left(\frac{n\pi}{a}\right)x\,dx$

$= \frac{2}{n\pi c}\left[-\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}\right)x\right]_0^a$

$= \frac{2a}{n^2\pi^2 c}\left[-\cos(n\pi)+1\right]$

$\Rightarrow b_n = \frac{2a}{n^2\pi^2 c}(1-\cos n\pi).$

(9) $\Rightarrow u(x,t) = \sum_n \sin\left(\frac{n\pi}{a}\right)x\left\{0 + \frac{2a}{n^2\pi^2 c}(1-\cos n\pi)\sin\left(\frac{n\pi c}{a}\right)t\right\}$

$\Rightarrow u(x,t) = \sum_n \frac{2a}{n^2\pi^2 c}(1-\cos n\pi)\sin\left(\frac{n\pi}{a}\right)x \sin\left(\frac{n\pi c}{a}\right)t.$

This is the required solution.

Example 4:

Solve the following homogeneous problem

$u_{tt} = c^2 u_{xx}, \quad 0<x<\pi, \ t>0.$

$u(x,0) = 0, \quad u_t(x,0) = 8\sin^2 x.$

$u(0,t) = 0, \quad u(\pi,t) = 0.$

Solution:

Given that $u_{tt} = c^2 u_{xx}, \quad 0<x<\pi$

$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ ——— (1)

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$u(0,t)=0, \quad u(\pi,t)=0$ ← (2)

$u(x,0)=0, \quad u_t(x,0)=8\sin^2 x$ ——— (3)

By example 2 page 198 we can write

(9) $u(x,t)=\sum_n \sin(nx)\{a_n\cos(nc)t+b_n\sin(nc)t\}$ ——— (9)

The initial conditions give us

(3) $\Rightarrow \quad u(x,0)=0 \quad =f(x)$ (say)

and $u_t(x,0) = 8\sin^2 x$ (say).

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(10) $\Rightarrow a_n = 0$

and $b_n = \frac{2}{nc}\int_0^{\pi} 8\sin^2 x \sin(nx)\,dx.$

Example 5:

Solve the problem

$u_{tt}+au_t+bu=c^2u_{xx}, \quad 0<x<l, \quad t>0$

$u(x,0)=f(x), \quad u_t(x,0)=0$

$u(0,t)=0=u(l,t).$

Solution.

Given that $u_{tt}+au_t+bu=c^2u_{xx}, \quad 0<x<l$ ——— (1)

(I. Conds) $u(x,0)=f(x), \quad u_t(x,0)=0$ ——— (2)

(B. Cs) $u(0,t)=0, \quad u(l,t)=0$ ——— (3)

Therefore,

(1) $\Rightarrow \frac{\partial^2 u}{\partial t^2}+a\frac{\partial u}{\partial t}+bu=c^2\frac{\partial^2 u}{\partial x^2}$

we solve this problem by separation of variables method

Let $u(x,t)=X(x)T(t)$ ——— (4)

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$$\Rightarrow \frac{\partial^2 X\bar{T}}{\partial t^2} + a\frac{\partial X\bar{T}}{\partial t} + bX\bar{T} = c^2\frac{\partial^2 X\bar{T}}{\partial x^2}$$

$$\Rightarrow X\frac{\partial^2 \bar{T}}{\partial t^2} + aX\frac{\partial \bar{T}}{\partial t} + bX\bar{T} = c^2\bar{T}\frac{\partial^2 X}{\partial x^2}$$

$$\Rightarrow X\frac{d^2\bar{T}}{dt^2} + aX\frac{d\bar{T}}{dt} + bX\bar{T} = c^2\bar{T}\frac{d^2X}{dx^2}$$

$$\Rightarrow X\left(\frac{d^2\bar{T}}{dt^2} + a\frac{d\bar{T}}{dt} + b\bar{T}\right) = c^2\bar{T}\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{\bar{T}}\left(\frac{d^2\bar{T}}{dt^2} + a\frac{d\bar{T}}{dt} + b\bar{T}\right) = \frac{c^2}{X}\frac{d^2X}{dx^2} \quad \text{---(5)}$$

Since the L.H.S of (5) is function of t only and the R.H.S of eq (5) is function of x only, both sides are equal if they are equal to some constant say $-\lambda^2$.

$$(5) \Rightarrow \frac{1}{\bar{T}}\left(\frac{d^2\bar{T}}{dt^2} + a\frac{d\bar{T}}{dt} + b\bar{T}\right) = -\lambda^2$$

and $$\frac{c^2}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2\bar{T}}{dt^2} + a\frac{d\bar{T}}{dt} + (b+\lambda^2)\bar{T} = 0 \quad \text{---(6)}$$

and $$c^2\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{---(7)}$$

$$(7) \Rightarrow c^2D^2X + \lambda^2 X = 0 \qquad \because D^2 = d^2/dx^2$$

$$\Rightarrow (c^2D^2 + \lambda^2)X = 0$$

The characteristic equation is

$$c^2D^2 + \lambda^2 = 0.$$

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$\Rightarrow D^2 = -\lambda^2/c^2 \quad \Rightarrow D = \pm i\lambda/c.$

$\Rightarrow X(x) = C_1 \cos(\frac{\lambda}{c})x + C_2 \sin(\frac{\lambda}{c})x$ ——— (8).

(3) implies

$u(0,t) = X(0)\bar{T}(t) = 0 \quad \Rightarrow \quad X(0) = 0$

and $u(l,t) = X(l)\bar{T}(t) = 0 \quad \Rightarrow \quad X(l) = 0.$

(8) $\Rightarrow \quad X(0) = C_1 \quad \Rightarrow \boxed{C_1 = 0}$

$X(l) = C_2 \sin(\frac{\lambda}{c} l) = 0.$

For non trivial solution, we choose $C_2 \neq 0$.

$\Rightarrow \sin\frac{\lambda l}{c} = 0$

$\Rightarrow \frac{\lambda l}{c} = n\pi \quad ; n = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \frac{\lambda n}{c} = \frac{n\pi}{l} \quad \Rightarrow \lambda_n = \frac{n\pi c}{l}, \quad n = 1, 2, 3 \cdots$

(8) $\Rightarrow X_n(x) = C_n \sin(\frac{\lambda_n}{c})x$

$\Rightarrow X_n(x) = C_n \sin(\frac{n\pi}{l})x \quad , n = 1, 2, 3, \cdots$

$\Delta^2 = d^2/dt^2$

(6) $\Rightarrow \Delta^2\bar{T} + a\Delta\bar{T} + (b+\lambda^2)\bar{T} = 0 \quad \because \Delta = d/dt$

$\Rightarrow (\Delta^2 + a\Delta + (b+\lambda^2))\bar{T} = 0$

The auxiliary equation is

$\Delta^2 + a\Delta + (b+\lambda^2) = 0.$

which is quadratic equation in Δ.

$\Rightarrow \Delta = \frac{-a \pm \sqrt{a^2 - 4(b+\lambda^2)}}{2}$

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Here we choose the case $a^2-4(b+\lambda^2)<0$ for non trivial solution and ignore the other two cases that is $a^2-4(b+\lambda^2)=0$ and $a^2-4(b+\lambda^2)>0$ because they give trivial solutions.

So $a^2-4(b+\lambda^2)<0$

$$\Rightarrow \Delta = \frac{-a \pm \sqrt{4(b+\lambda^2)-a^2}\, i}{2}$$

or $$\Delta = \frac{-a \pm \sqrt{4(b+\lambda_n^2)-a^2}\, i}{2} \quad \text{(total)}$$

where $\lambda_n = \frac{n\pi c}{\ell}$, $n=1,2,3,\cdots$

or $$\Delta = \frac{-a \pm 2Ai}{2} = \frac{-a}{2} \pm Ai \quad \sqrt{4(b+\lambda^2)-a^2} = 2A \text{ (say)}$$

$$\Rightarrow T(t) = (C_3 \cos At + C_4 \sin At)\, e^{-a/2\, t}$$

The eigensolutions are

$$T_n(t) = e^{-a/2\, t}(C_n' \cos A_n t + C_n^* \sin A_n t).$$

(4) $$\Rightarrow u_n(x,t) = X_n(x)\, T_n(t).$$

$$= C_n \sin\left(\frac{n\pi}{\ell}\right) x \left\{C_n' \cos A_n t + C_n^* \sin A_n t\right\} e^{-\frac{at}{2}}$$

$$= e^{-a/2\, t} \sin\left(\frac{n\pi}{\ell}\right) x \left\{a_n' \cos A_n t + b_n' \sin A_n t\right\}.$$

where $a_n' = C_n C_n^*$, $b_n' = C_n C_n^*$

By the principle of superposition, the linear combination of the above solution

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is also a solution.

i.e. $u(x,t) = \sum_n d_n u_n(x,t)$

$$= \sum_n d_n e^{-\frac{at}{2}} \sin\left(\frac{n\pi}{\ell}\right)x \{a_n' \cos A_n t + b_n' \sin A_n t\}.$$

$$\Rightarrow u(x,t) = \sum_n \sin\left(\frac{n\pi}{\ell}\right)x\, e^{-\frac{at}{2}} \{a_n \cos A_n t + b_n \sin A_n t\} \longrightarrow (9)$$

where $a_n = d_n a_n'$, $b_n = d_n b_n'$.

The initial conditions give us.

$(2) \Rightarrow u(x,0) = f(x), \quad u_t(x,0) = 0$

$$(9) \Rightarrow u(x,0) = \sum_n a_n \sin\left(\frac{n\pi}{\ell}\right)x = f(x)$$

$$\Rightarrow a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi}{\ell}\right)x\, dx. \longrightarrow (10)$$

and $u_t(x,0) = \sum_n \Big\{ \sin\left(\frac{n\pi}{\ell}\right)x \Big[e^{-\frac{a}{2}t}\left(-\frac{a}{2}\right)\{a_n \cos A_n t + b_n \sin A_n t\}$

$$+ e^{-\frac{a}{2}t}\{-a_n A_n \sin A_n t + b_n A_n \cos A_n t\}\Big]\Big|_{t=0}$$

$$\Rightarrow u_t(x,0) = \sum_n \sin\left(\frac{n\pi}{\ell}\right)x \left\{-\frac{a}{2}(a_n) + b_n A_n\right\} = 0$$

$$= \sum_n \sin\left(\frac{n\pi}{\ell}\right)x \left\{-\frac{a a_n}{2} + b_n A_n\right\} = 0 = g(x)$$

$$\Rightarrow -\frac{a}{2}a_n + b_n A_n = \frac{2}{\ell}\int_0^{\ell} g(x) \sin\left(\frac{n\pi}{\ell}x\right) dx$$

$$\Rightarrow -\frac{a}{2}a_n + b_n A_n = 0 \quad \because g(x) = 0$$

$$\Rightarrow b_n A_n = \frac{a}{2} a_n$$

$$\Rightarrow b_n = \frac{a}{2A_n} a_n$$

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using (10)

$$\Rightarrow b_n = \frac{a}{\sqrt{4(b+\lambda_n^2)-a^2}} \cdot \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}\right)x\,dx$$

$$= \frac{2a}{\ell\sqrt{4\left(b+\frac{n^2\pi^2c^2}{\ell^2}\right)-a^2}} \cdot \int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}\right)x\,dx$$

$$= \frac{2a}{\sqrt{4(b\ell^2+n^2\pi^2c^2)-a^2\ell^2}} \int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}\right)x\,dx$$

where $n = 1, 2, 3, \ldots$

[Exercise]

Q1. Find the solution of the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \quad 0<x<\pi, \quad t>0$$

$$u(0,t) = u(\pi,t) = 0$$

$$u(x,0) = \sin 3x - 4\sin 10x, \quad t\geq 0.$$

$$u_t(x,0) = 2\sin 4x + \sin 6x.$$

Solution.

Given that $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$, $0<x<\pi$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{2^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$

$$u(0,t) = 0, \quad u(\pi,t) = 0 \quad \text{——— (2)}$$

$$u(x,0) = \sin 3x - 4\sin 10x$$

$$u_t(x,0) = 2\sin 4x + \sin 6x \quad \text{——— (3)}$$

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By example 2, page 198.

⑨ $\Rightarrow u(x,t) = \sum_n \sin(nx)\{a_n \cos(2n)t + b_n \sin(2n)t\}$.

where $a_n = \frac{2}{\pi}\int_0^{\pi} (\sin 3x - 4\sin 10x)\sin(nx)\,dx$ by ⑩

and $b_n = \frac{2}{n\pi}\int_0^{\pi} (2\sin 4x + \sin 6x)\sin(n)x\,dx$ by ⑪

Q2:

Solve the following vibrating string problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + K\frac{\partial u}{\partial t}, \quad 0 < x < a, \; t > 0.$$

with boundary and initial conditions given by

$u(0,t) = 0$, $u(a,t) = 0$ where K is small and +ive.

and $u(x,0) = 0$, $u_t(x,0) = 1$.

Solution.

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + K\frac{\partial u}{\partial t}$ ——— ①

$u(0,t) = 0$, $u(a,t) = 0$ ——— ②

$u(x,0) = 0$, $u_t(x,0) = 1$ ——— ③

To solve this problem we use method of separation of variables.

Let $u(x,t) = X(x)T(t)$ ——— ④

Then ① $\Rightarrow \frac{\partial^2 XT}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 XT}{\partial t^2} + K\frac{\partial XT}{\partial t}$

$\Rightarrow T\frac{d^2X}{dx^2} = \frac{1}{c^2}X\frac{d^2T}{dt^2} + KX\frac{dT}{dt}$

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$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\left(\frac{1}{c^2}\frac{d^2T}{dt^2} + K\frac{dT}{dt}\right) \quad \text{——— (5)}$$

Since the L.H.S of equation (5) is function of x only and the R.H.S is function of t only, both are equal if they are equal to some constant say $-\lambda^2$.

i.e. $$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

and $$\frac{1}{T}\left(\frac{1}{c^2}\frac{d^2T}{dt^2} + K\frac{dT}{dt}\right) = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{——— (6)}$$

and $$\frac{d^2T}{dt^2} + Kc^2\frac{dT}{dt} + \lambda^2 c^2 T = 0 \quad \text{——— (7)}$$

(6) $\Rightarrow D^2X + \lambda^2 X = 0 \qquad \because D^2 = \frac{d}{dx^2}$

$\Rightarrow (D^2 + \lambda^2)X = 0$

$\Rightarrow D^2 + \lambda^2 = 0$ auxiliary equation.

$\Rightarrow D = \pm i\lambda$

$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x \quad \text{——— (8)}$

(2) $\Rightarrow u(0,t) = X(0)T(t) = 0$

$\Rightarrow X(0) = 0$

$u(a,t) = X(a)T(t) = 0$

$\Rightarrow X(a) = 0.$

(8) $\Rightarrow X(0) = C_1 \Rightarrow \boxed{C_1 = 0}$

$\Rightarrow X(x) = C_2 \sin\lambda x.$

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$X(a) = C_2 \sin\lambda a = 0.$

For non trivial solution we choose $C_2 \neq 0$.

$\Rightarrow \sin\lambda a = 0.$

$\Rightarrow \lambda a = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots$

$\Rightarrow \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3 \cdots$

(8) $\Rightarrow X_n(x) = C_n \sin\left(\frac{n\pi}{a}\right)x, \quad n = 1, 2, 3, \cdots$

(7) $\Rightarrow \Delta^2 \bar{T} + Kc^2 \Delta \bar{T} + \lambda^2 c^2 \bar{T} = 0, \quad \Delta = \frac{d}{dt}, \Delta^2 = \frac{d^2}{dt^2}$

$\Rightarrow (\Delta^2 + Kc^2\Delta + \lambda^2 c^2)\bar{T} = 0$

The characteristic equation is

$\Delta^2 + Kc^2\Delta + \lambda^2 c^2 = 0.$

$$\Rightarrow \Delta = \frac{-Kc^2 \pm \sqrt{K^2c^4 - 4\lambda^2 c^2}}{2}$$

Here we choose the case that is $K^2c^4 - 4\lambda^2c^2 < 0$ for non trivial solution. The other two cases i.e. $K^2c^4 - 4\lambda^2c^2 = 0$ and $K^2c^4 - 4\lambda^2c^2 > 0$ give trivial solutions.

So $$\Delta = \frac{-Kc^2 \pm \sqrt{4\lambda^2c^2 - K^2c^4}\, i}{2}$$

$$= \frac{-Kc^2}{2} + \frac{\sqrt{4\lambda^2c^2 - K^2c^4}\, i}{2} = -\frac{Kc^2}{2} + Ai$$ (A (say))

$$\Rightarrow \bar{T}(t) = e^{\frac{-Kc^2}{2}t}\left(C_3 \cos At + C_4 \sin At\right).$$

$$\Rightarrow \bar{T}_n(t) = e^{\frac{-Kc^2}{2}t}\left(C_n' \cos A_n t + C_n^* \sin A_n t\right).$$

(4) $\Rightarrow u_n(x,t) = X_n(x)\,\bar{T}_n(t).$

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$\Rightarrow u_n(x,t) = C_n \sin(\frac{n\pi}{a})x \{C_n' \cos A_n t + C_n^* \sin A_n t) e^{-\frac{kc^2}{2}t}$

$\Rightarrow u_n(x,t) = \sin(\frac{n\pi}{a})x \{a_n' \cos A_n t + b_n' \sin A_n t) e^{-\frac{kc^2}{2}t}$

where $a_n' = C_n C_n'$, $b_n' = C_n C_n^*$.

By the principle of superposition, the linear combination of above solution is also a solution.

i.e. $u(x,t) = \sum_n d_n u_n(x,t)$.

$\Rightarrow u(x,t) = \sum_n \sin(\frac{n\pi}{a})x \{a_n \cos A_n t + b_n \sin A_n t\} e^{-\frac{kc^2}{2}t}$ — (9)

where $a_n = d_n a_n'$, $b_n = d_n b_n'$.

The initial conditions are given by

$u(x,0) = 0 = f(x)$. (say)

$u_t(x,0) = 1 = g(x)$ (say).

(9) $\Rightarrow u(x,0) = \sum_n a_n \sin(\frac{n\pi}{a})x = f(x)$.

$\Rightarrow a_n = \frac{2}{a}\int_0^a f(x) \sin(\frac{n\pi}{a})x \, dx$

$\Rightarrow a_n = 0 \quad \because f(x) = 0$.

(9) $\Rightarrow u_t(x,0) = \sum_n \sin(\frac{n\pi}{a})x \Big[e^{-\frac{kc^2}{2}t}(-\frac{kc^2}{2})(a_n \cos A_n t + b_n \sin A_n t) + e^{-\frac{kc^2}{2}t}(-a_n A_n \sin A_n t + b_n A_n \cos A_n t)\Big|_{t=0}$

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$$\Rightarrow u_t(x,0) = \sum_n \sin\left(\frac{n\pi}{a}\right)\left(-\frac{kc^2}{2}a_n + b_n A_n\right) = g(x).$$

$$\Rightarrow b_n A_n - \frac{kc^2}{2}a_n = \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}\right)x\,dx.$$

$$\Rightarrow b_n A_n = \frac{2}{a}\int_0^a \sin\left(\frac{n\pi}{a}\right)x\,dx \;;\; a_n = 0,\ g(x) = 1.$$

$$\Rightarrow b_n = \frac{2}{aA_n}\int_0^a \sin\left(\frac{n\pi}{a}\right)x\,dx.$$

$$= \frac{2}{aA_n}\left[\frac{-a}{n\pi}\cos\left(\frac{n\pi}{a}\right)x\right]_0^a = \frac{2}{aA_n}\left(-\frac{a}{n\pi}\cos n\pi + \frac{a}{n\pi}\right)$$

$$= \frac{2}{n\pi A_n}(1-\cos n\pi).$$

$$= \frac{2}{n\pi\sqrt{4\lambda_n^2c^2 - k^2c^4}/2}(1-\cos n\pi).$$

$$= \frac{4}{n\pi\sqrt{\frac{4n^2\pi^2}{a^2}c^2 - k^2c^4}}(1-\cos n\pi)$$

$$b_n = \frac{4a(1-\cos n\pi)}{n\pi\sqrt{4n^2\pi^2c^2 - k^2a^2c^4}}$$

Thus

$$(9) \Rightarrow u(x,t) = \sum_n \frac{4a(1-\cos n\pi)}{n\pi\sqrt{4n^2\pi^2c^2 - k^2a^2c^4}}\sin\left(\frac{n\pi}{a}\right)x\,\sin A_n t\, e^{-\frac{kc^2}{2}t}$$

where $A_n = \frac{\sqrt{4\lambda_n^2c^2 - k^2c^4}}{2}$ which is req. solution.

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Question 3:

Solve the following problem associated with the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1$$

Subject to the initial and boundary condition.

$$u(x,0) = x(1-x), \quad u_t(x,0) = 0$$
$$u(0,t) = 0, \quad u(1,t) = 0.$$

Solution.

Given that $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$

Subject to $u(x,0) = x(1-x)$, $u_t(x,0) = 0$ ——— (2)

$u(0,t) = 0$, $u(1,t) = 0$ ——— (3)

We solve this problem by separation of variables method.

Let $u(x,t) = X(x)T(t)$ ——— (4)

{See example (2) page 198.}

Then

$$u(x,t) = \sum_n \sin(n\pi)x\{a_n \cos(n\pi c)t + b_n \sin(n\pi c)t\} \rightarrow (9)$$

The initial conditions give us

(2) $\Rightarrow u(x,0) = x(1-x) = f(x)$ (say)

and $u_t(x,0) = 0 = g(x)$ (say).

$$(10) \Rightarrow a_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi)x\,dx$$

{See page 202}

$$= 2\int_0^1 x(1-x)\sin(n\pi)x\,dx$$

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$$= 2\int_0^1 (x^2 - x^2)\sin(n\pi)x\,dx$$

$$= 2\left[-(x-\tfrac{1}{2})\frac{\cos(n\pi)x}{n\pi}\Big|_0^1 + \int_0^1 \frac{\cos(n\pi)x}{n\pi}(1-2x)\,dx\right]$$

$$= 2\left[0 + \frac{1}{n\pi}\int_0^1 \cos(n\pi)x\,dx - \frac{2}{n\pi}\int_0^1 x\cos(n\pi)x\,dx\right]$$

$$= \frac{2}{n\pi}\left[\frac{\sin(n\pi)x}{n\pi}\Big|_0^1 - 2\left\{x\frac{\sin(n\pi)x}{n\pi}\Big|_0^1 - \int_0^1 \frac{\sin(n\pi)x}{n\pi}dx\right\}\right]$$

$$= \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi)x\,dx = \frac{4}{n^2\pi^2}\left[-\frac{\cos(n\pi)x}{n\pi}\right]_0^1$$

$$= \frac{4}{n^3\pi^3}(-\cos n\pi + 1)$$

$$\Rightarrow a_n = \frac{4}{n^3\pi^3}(1-\cos n\pi).$$

(11) $\Rightarrow b_n = \frac{2}{n\pi c}\int_0^1 g(x)\sin(\frac{n\pi}{a})x\,dx$

$\Rightarrow b_n = 0 \quad \because g(x) = 0.$

Hence the solution is

(9) $\Rightarrow u(x,t) = \sum_n \frac{4}{n^3\pi^3}(1-\cos n\pi)\sin(n\pi)x\cos(n\pi c)t$

Q4: 2006 Solve the following problem.

$u_{xx} = u_{tt} + u, \quad 0 < x < \pi$

$u(0,t) = 0, \quad u(\pi,t) = 0.$

$u(x,0) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}, \quad u_t(x,0) = 0$

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Solution.

Given that $u_{xx} = u_{tt} + u$ ——— (1)

$u(0,t) = 0, \quad u(\pi,t) = 0$ ——— (2)

$$u(x,0) = \begin{cases} x & \text{if } 0 \le x \le \pi/2 \\ \pi - x & \text{if } \frac{\pi}{2} < x \le \pi, \end{cases} \quad u_t(x,0) = 0 \longrightarrow (3)$$

(1) $\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} + u$.

We solve this problem by separation of variables method.

Let $u(x,t) = X(x)T(t)$ ——— (4)

$$\frac{\partial^2 XT}{\partial x^2} = \frac{\partial^2 XT}{\partial t^2} + XT$$

$$\Rightarrow T\frac{d^2X}{dx^2} = X\frac{d^2T}{dt^2} + XT$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\left(\frac{d^2T}{dt^2} + T\right) \text{ ——— (5)}$$

Since the L.H.S of (5) is function of x only and the R.H.S is function of t only, both sides are equal only if they are equal to some constant say $-\lambda^2$.

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\text{and} \quad \frac{1}{T}\left(\frac{d^2T}{dt^2} + T\right) = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \text{ ——— (6)}$$

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and $\frac{d^2T}{dt^2} + (1+\lambda^2)T = 0$. ——— (7)

(6) $\Rightarrow X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$ ——— (8)

(2) $\Rightarrow X(0) = 0, \quad X(\pi) = 0$.

(8) & $\Rightarrow X_n(x) = C_n \sin\lambda_n x$,

$X_n(\pi) = 0$
$\Rightarrow C_n \sin\lambda_n \pi = 0$
$\Rightarrow \lambda_n \pi = n\pi$
$\boxed{\lambda_n = n}$

$X_n(x) = C_n \sin(n)x \quad ; n = 1, 2, 3 \cdots$

(7) $\Rightarrow \Delta^2 T + (1+\lambda^2)T = 0 \quad \because \Delta^2 = d^2/dt^2$.

$\Rightarrow (\Delta^2 + (1+\lambda^2))T = 0$.

The char. equation is

$\Delta^2 + (1+\lambda^2) = 0$.

$\Rightarrow \Delta^2 = -(1+\lambda^2)$

$\Rightarrow \Delta = \pm i\sqrt{1+\lambda^2}$

$\Rightarrow \bar{T}(t) = C_3 \cos\sqrt{1+\lambda^2}\, t + C_4 \sin\sqrt{1+\lambda^2}\, t$

$\bar{T}_n(t) = C_n' \cos(\sqrt{1+\lambda_n^2})t + C_n^* \sin(\sqrt{1+\lambda_n^2})t$

Thus

$$u(x,t) = \sum_n \sin nx \{a_n \cos\sqrt{1+n^2}\, t + b_n \sin\sqrt{1+n^2}\, t\}.$$ ——— (9)

The initial conditions will give us.

$$u(x,0) = \begin{cases} x & \text{if } 0 \le x \le \frac{\pi}{2} \\ \pi - x & \text{if } \frac{\pi}{2} \le x \le \pi \end{cases} = f(x) \text{ (say)}.$$

$u_t(x,0) = 0 = g(x)$ (say).

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(10) $\Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(n)x\,dx$ ——— (10)

$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx\right]$

$\Rightarrow a_n = \frac{2}{\pi}(I_1 + I_2)$ ——— (10*)

where $I_1 = \int_0^{\pi/2} x\sin nx\,dx$

Integrating by parts, we get

$I_1 = -x\frac{\cos nx}{n}\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos nx}{n}\,dx$

$= \frac{1}{n}\left(-x\cos nx + \frac{1}{n}\sin nx\right)\Big|_0^{\pi/2}$

$= \frac{1}{n}\left(-\frac{\pi}{2}\cos n\frac{\pi}{2} + 0 + \frac{1}{n}\sin n\frac{\pi}{2} - 0\right)$

$I_1 = \frac{1}{n^2}\sin n\frac{\pi}{2}$

and $I_2 = \int_{\pi/2}^{\pi}(\pi - x)\sin nx\,dx$

$= -(\pi - x)\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} -\frac{\cos nx}{n}\,dx$

$= \frac{1}{n}\left[-(\pi - x)\cos nx - \frac{1}{n}\sin nx\right]_{\pi/2}^{\pi}$

$= \frac{1}{n}\left[0 + \frac{\pi}{2}\cos n\frac{\pi}{2} - \frac{1}{n}\sin n\pi + \frac{1}{n}\sin n\frac{\pi}{2}\right]$

$\Rightarrow I_2 = \frac{1}{n^2}\sin n\frac{\pi}{2}$.

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Thus equation (10)* becomes

$$C_n = \frac{2}{\pi}\left(\frac{2}{n^2}\sin n\frac{\pi}{2}\right) = \frac{4}{n^2\pi}\sin n\frac{\pi}{2}$$

where $n = 1, 2, 3, \cdots$

and $b_n = 0 \quad \because g(x) = 0.$

Q5:

The displacements $u(x,t)$ of a uniform thin beam satisfy

$$\frac{\partial^4 u}{\partial x^4} \overset{(-ve)}{-} \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < a, \; t > 0.$$

If the beam is simply supported at the ends, the boundary conditions are

$$u(0,t) = 0, \quad u(a,t) = 0$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2}{\partial x^2}u(a,t) = 0.$$

find product solutions to this problem.

Solution.

Given that $\frac{\partial^4 u}{\partial x^4} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \; 0 < x < a$ ——— (1)

$$u(0,t) = 0, \; u(a,t) = 0 \quad \text{——— (2)}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \; \frac{\partial^2}{\partial x^2}u(a,t) = 0 \quad \text{——— (3)}$$

we solve this problem by method of Separation of variables.

Let $u(x,t) = X(x)T(t)$ ——— (4)

(1) $\Rightarrow \frac{\partial^4 XT}{\partial x^4} = \frac{1}{c^2}\frac{\partial^2 XT}{\partial t^2}$

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$$\Rightarrow T\frac{d^4X}{dx^4} = \frac{1}{c^2}X\frac{d^2T}{dt^2}$$

$$\Rightarrow \frac{1}{X}\frac{d^4X}{dx^4} = \frac{1}{c^2T}\frac{d^2T}{dt^2} \quad \text{——— (5)}$$

Since the L.H.S of (5) is function of x only and the R.H.S of (5) is function of t only, both sides are equal only if they are equal to some constant say λ^2.

$$\Rightarrow \frac{1}{X}\frac{d^4X}{dx^4} = \lambda^2$$

and $$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \lambda^2$$

$$\Rightarrow \frac{d^4X}{dx^4} - \lambda^2 X = 0 \quad \text{——— (6)}$$

and $$\frac{d^2T}{dt^2} - \lambda^2 c^2 T = 0 \quad \text{——— (7)}$$

(6) $\Rightarrow D^4X - \lambda^2 X = 0 \quad \because D^4 = d^4/dx^4$.

$\Rightarrow (D^4 - \lambda^2)X = 0$.

$\Rightarrow D^4 - \lambda^2 = 0$

$\Rightarrow (D^2)^2 - (\lambda)^2 = 0$ for $\lambda > 0$.

$\Rightarrow (D^2 + \lambda)(D^2 - \lambda) = 0$

$\Rightarrow D^2 + \lambda = 0$ or $D^2 - \lambda = 0$.

$\Rightarrow D^2 = -\lambda$, $D^2 = \lambda$.

$\Rightarrow D = \pm i\sqrt{\lambda}$, $D = \pm\sqrt{\lambda}$

$\Rightarrow D = \pm\sqrt{\lambda}, \pm i\sqrt{\lambda}$.

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$$\Rightarrow X(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x} + C_3 \cos\sqrt{\lambda}x + C_4 \sin\sqrt{\lambda}x \quad \text{—— (8)}$$

Applying boundary conditions, we get,

$$X(0) = C_1 + C_2 + C_3 = 0$$

$$\Rightarrow C_1 + C_2 + C_3 = 0 \quad \text{—— (9)}$$

$$X(a) = 0 \Rightarrow C_1 e^{\sqrt{\lambda}a} + C_2 e^{-\sqrt{\lambda}a} + C_3 \cos\sqrt{\lambda}a + C_4 \sin\sqrt{\lambda}a = 0 \quad \text{—— (10)}$$

(3)

$$\frac{\partial^2 u(x,t)}{\partial x^2} = T(t)\frac{\partial^2 X}{\partial x^2} = T(t)\frac{d^2X}{dx^2}$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = 0 \Rightarrow T(t)\frac{d^2X(0)}{dx^2} = 0$$

$$\Rightarrow \frac{d^2X(0)}{dx^2} = 0. \quad \text{and similarly } \frac{d^2X(a)}{dx^2} = 0.$$

$$\frac{dX(x)}{dx} = C_1\sqrt{\lambda}e^{\sqrt{\lambda}x} - C_2\sqrt{\lambda}e^{-\sqrt{\lambda}x} - C_3\sqrt{\lambda}\sin\sqrt{\lambda}x + C_4\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$\frac{d^2X(x)}{dx^2} = C_1\lambda e^{\sqrt{\lambda}x} + C_2\lambda e^{-\sqrt{\lambda}x} - C_3\lambda\cos\sqrt{\lambda}x - C_4\lambda\sin\sqrt{\lambda}x$$

$$\frac{d^2X(0)}{dx^2} = C_1\lambda + C_2\lambda - C_3\lambda = 0$$

$$\Rightarrow C_1\lambda + C_2\lambda - C_3\lambda = 0.$$

$$\Rightarrow \lambda(C_1 + C_2 - C_3) = 0.$$

$$\Rightarrow C_1 + C_2 - C_3 = 0 \quad \text{—— (11)} \quad \because \lambda \neq 0.$$

$$\frac{d^2X(a)}{dx^2} = C_1\lambda e^{\sqrt{\lambda}a} + C_2\lambda e^{-\sqrt{\lambda}a} - C_3\lambda\cos\sqrt{\lambda}a - C_4\lambda\sin\sqrt{\lambda}a$$

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$$\Rightarrow \lambda(C_1 e^{\sqrt{\lambda}a} + C_2 e^{-\sqrt{\lambda}a} - C_3 \cos\sqrt{\lambda}a - C_4 \sin\sqrt{\lambda}a) = 0$$

$$\Rightarrow C_1 e^{\sqrt{\lambda}a} + C_2 e^{-\sqrt{\lambda}a} - C_3 \cos\sqrt{\lambda}a - C_4 \sin\sqrt{\lambda}a = 0 \quad \because \lambda \neq 0 \quad \text{---(12)}$$

Subtracting (9) and (11), we get

$$C_1 + C_2 + C_3 = 0$$
$$+C_1 + C_2 - C_3 = 0 \quad \Rightarrow \boxed{C_3 = 0}$$
$$2C_3 = 0$$

(9) $\Rightarrow C_1 = -C_2$.

(10) $\Rightarrow -C_2 e^{\sqrt{\lambda}a} + C_2 e^{-\sqrt{\lambda}a} + C_4 \sin\sqrt{\lambda}a = 0$ ---(13)

(12) $\Rightarrow -C_2 e^{\sqrt{\lambda}a} + C_2 e^{-\sqrt{\lambda}a} - C_4 \sin\sqrt{\lambda}a = 0$ ---(14)

Adding (13) and (14), we get.

$$-2C_2 e^{\sqrt{\lambda}a} + 2C_2 e^{-\sqrt{\lambda}a} = 0$$
$$\Rightarrow 2C_2(e^{-\sqrt{\lambda}a} - e^{\sqrt{\lambda}a}) = 0$$

$\Rightarrow \boxed{C_2 = 0}$ because $2(e^{-\sqrt{\lambda}a} - e^{\sqrt{\lambda}a}) \neq 0$.

So $\boxed{C_1 = 0}$

(13) Subtracting (13) and (14), we get,

$$2C_4 \sin\sqrt{\lambda}a = 0.$$

For non trivial solution, we choose

$C_4 \neq 0$.

$$\Rightarrow \sin\sqrt{\lambda}a = 0.$$
$$\sqrt{\lambda}a = n\pi \quad , n = 0, \pm 1, \pm 2, \cdots$$
$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{a}.$$

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$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{a^2} \quad ; n=1,2,3,\cdots$$

(8) $\Rightarrow X_n(x) = C_n \sin\sqrt{\lambda_n}\, x \quad ; n=1,2,3\cdots$

$\Rightarrow X_n(x) = C_n \sin\left(\frac{n\pi}{a}\right)x \quad ; n=1,2,3\cdots$

(7) $\Rightarrow \Delta^2 T - \lambda^2 c^2 T = 0 \qquad \because \Delta^2 = d^2/dt^2$

$\Rightarrow (\Delta^2 - \lambda^2 c^2) T = 0$

The charac. equation is

$\Delta^2 - \lambda^2 c^2 = 0$

$\Rightarrow \Delta = \lambda c$

$\Rightarrow T(t) = C_5 \cos(\lambda c)t + C_6 \sin(\lambda c)t$

or $T_n(t) = C_n' \cos(\lambda_n c)t + C_n^* \sin(\lambda_n c)t$

$\Rightarrow T_n(t) = C_n' \cos\left(\frac{n^2\pi^2}{a^2}c\right)t + C_n^* \sin\left(\frac{n^2\pi^2}{a^2}c\right)t$

Thus the product of solutions is

$$u(x,t) = \sin\left(\frac{n\pi}{a}\right)x\left\{A_n \cos\left(\frac{n^2\pi^2}{a^2}c\right)t + B_n \sin\left(\frac{n^2\pi^2}{a^2}c\right)t\right\}$$

where $A_n = C_n C_n'$; $B_n = C_n C_n^*$

$n=1,2,3,\cdots$

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Nonhomogeneous problems.

Example 1.

Solve the problem

$$u_{tt} = u_{xx} + h, \quad 0 \le x \le 1.$$

$$u(x,0) = x(1-x), \quad u_t(x,0) = 0$$

$$u(0,t) = 0, \quad u(1,t) = 0.$$

where h is a constant.

Solution.

Given that $u_{tt} = u_{xx} + h$ ——— (1)

$u(0,t) = 0, \quad u(1,t) = 0$ ——— (2)

and $u(x,0) = x(1-x), \quad u_t(x,0) = 0$ ——— (3)

Suppose $u(x,t) = v(x,t) + w(x)$ ——— (4)

is the solution of given PDE.

Then

$$(1) \Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} + \frac{\partial^2 w(x)}{\partial t^2}^{0} = \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 w(x)}{\partial x^2} + h$$

$$\Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} = \frac{\partial^2 v(x,t)}{\partial x^2} + \frac{d^2 w(x)}{dx^2} + h$$

$$\Rightarrow v_{tt} = v_{xx} + w'' + h, \quad 0 \le x \le 1$$

——— (5)

$$(2) \Rightarrow v(0,t) + w(0) = 0, \quad v(1,t) + w(1) = 0$$

$$\Rightarrow \left.\begin{aligned} v(0,t) &= 0, & v(1,t) &= 0 \\ \text{and } w(0) &= 0 & w(1) &= 0 \end{aligned}\right\} \text{ ——— (6)}$$

$$(3) \Rightarrow v(x,0) + w(x) = x(1-x), \quad v_t(x,0) + w_t(x)^{0} = 0$$

$$\Rightarrow v(x,0) = x(1-x) - w(x), \quad v_t(x,0) = 0 \text{ ——— (7)}$$

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The diff. equation ⑤, with the B.Cs and I.Conditions in ⑥ and ⑦, is equivalent to the following two problems.

$$V_{tt}(x,t) = V_{xx}(x,t) \longrightarrow ⑧ \quad ; \quad W''(x) + h = 0. \quad \text{——} ⑨$$

with

$$V(0,t) = 0, V(1,t) = 0 \quad ; \quad W(0) = 0, W(1) = 0$$

$$V(x,0) = x(1-x) - W(x)$$

$$V_t(x,0) = 0$$

Solving ⑨, we get

$$W''(x) = -h$$

$$\Rightarrow W'(x) = -hx + A \quad \text{where } A \text{ is constt of integ}$$

$$\Rightarrow W(x) = -\frac{hx^2}{2} + Ax + B \quad \text{——} ⑩$$

$$W(0) = B \Rightarrow \boxed{B = 0}$$

$$W(1) = -\frac{h}{2} + A + B = 0$$

$$\Rightarrow -\frac{h}{2} + A = 0 \quad \because B = 0$$

$$\Rightarrow \boxed{A = \frac{h}{2}}$$

$$⑩ \Rightarrow W(x) = -\frac{h}{2}x^2 + \frac{h}{2}x = \frac{h}{2}x(1-x) \quad \text{——} ⑪$$

$$⑧ \Rightarrow \frac{\partial^2 V(x,t)}{\partial t^2} = \frac{\partial^2 V(x,t)}{\partial x^2}$$

Let $V(x,t) = X(x)T(t)$

Then $X\frac{\partial^2 T}{\partial t^2} = T\frac{\partial^2 X}{\partial x^2}$

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$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} \quad \text{——— (12)}$$

$X(0) = 0$, $X(1) = 0$

Since the L.H.S of (12) is function of x only and the R.H.S is function of t only, both sides are equal only if they are equal to some constant say, $-\lambda^2$.

$$(12) \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \quad ; \quad \frac{1}{T}\frac{d^2T}{dt^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad ; \quad \frac{d^2T}{dt^2} + \lambda^2 T = 0$$

After solving these two equations, we get.

$$V(x,t) = \sum_n \sin(n\pi)x \{a_n \cos(n\pi)t + b_n \sin(n\pi)t\} \quad \text{——— (13)}$$

Applying initial conditions from equation (7), we have,

$$V(x,0) = \sum_n a_n \sin(n\pi)x = f(x) \text{ (say)}.$$

where $f(x) = x(1-x) - w(x) = V(x,0)$

$$= x(1-x) - \frac{h}{2}x(1-x)$$

$$= (1 - h/2)\, x(1-x).$$

So $a_n = \frac{2}{1}\int_0^1 f(x) \sin(n\pi)x \, dx$.

$$\Rightarrow a_n = 2\int_0^1 (1 - \tfrac{h}{2})\, x(1-x) \sin(n\pi)x \, dx.$$

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$$\Rightarrow a_n = (2-h)\int_0^1 x(1-x)\sin(n\pi)x\,dx.$$

and

$$V_t(x,t)\Big|_{t=0} = \sum_n \sin(n\pi)x\{-a_n(n\pi)\sin(n\pi)t + b_n(n\pi)\cos(n\pi)t\}_{t=0}$$

$$\Rightarrow V_t(x,0) = \sum_n \sin(n\pi)x\{b_n(n\pi)\} = g(x) \text{ (say)}.$$

$$\Rightarrow n\pi b_n = \frac{2}{1}\int_0^1 g(x)\sin(n\pi)x\,dx.$$

$$\Rightarrow b_n = 0 \quad \because g(x) = 0.$$

Thus by (4) the final solution is

$$u(x,t) = \sum_n a_n \sin(n\pi)x\cos(n\pi)t + \frac{h}{2}x(1-x)$$

Example 2.

Solve the problem

$$u_{tt} = c^2 u_{xx} + x^2, \quad 0 \le x \le 1.$$
$$u(x,0) = u_t(x,0) = 0$$
$$u(0,t) = 0 = u(1,t).$$

Solution.

Given that $u_{tt} = c^2 u_{xx} + x^2$, $0 \le x \le 1$ ——— (1)

$u(0,t) = 0$, $u(1,t) = 0$ ——— (2)

$u(x,0) = 0$, $u_t(x,0) = 0$ ——— (3)

Suppose $u(x,t) = v(x,t) + w(x)$ ——— (4)

is the solution of given PDE.

Then (1) $\Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} = c^2\left(\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 w(x)}{\partial x^2}\right) + x^2$

$$\Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} = c^2\frac{\partial^2 v(x,t)}{\partial x^2} + c^2\frac{d^2 w(x)}{dx^2} + x^2 \quad \text{——— (5)}$$

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② $\Rightarrow v(0,t) + w(0) = 0$, $v(1,t) + w(1) = 0$.

$\Rightarrow v(0,t) = 0$, $v(1,t) = 0$ and $w(0) = 0$, $w(1) = 0$ ——— ⑥

③ $\Rightarrow v(x,0) + w(x) = 0$, $v_t(x,0) + w_t(x) = 0$ ($w_t(x) \to 0$)

$\Rightarrow v(x,0) = -w(x)$, $v_t(x,0) = 0$ ——— ⑦

The DE ⑤, with B.Cs and I.Cs in ⑥ and ⑦, is equivalent to the following two problems.

$$\frac{\partial^2 v(x,t)}{\partial t^2} = c^2 \frac{\partial^2 v(x,t)}{\partial x^2} \quad ; \quad c^2 w''(x) + x^2 = 0 \quad \text{——— ⑨}$$

$$\Rightarrow \frac{\partial^2 v(x,t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 v(x,t)}{\partial t^2} \to ⑧ \qquad \text{with } w(0) = 0, \; w(1) = 0$$

with

$v(0,t) = 0$, $v(1,t) = 0$

$v(x,0) = -w(x) = f(x)$ (say)

$v_t(x,t) = 0 = g(x)$ (say).

Solving ⑨, we get

$$w''(x) = -\frac{x^2}{c^2}$$

integrating

$$\Rightarrow w'(x) = -\frac{x^3}{3c^2} + A \quad \text{where } A \text{ is const}^{\text{n}} \text{ of integration.}$$

$$\Rightarrow w(x) = -\frac{x^4}{12c^2} + Ax + B. \quad \text{——— ⑩}$$

$w(0) = B \Rightarrow \boxed{B = 0}$

$w(1) = \frac{-1}{12c^2} + A = 0$

$\Rightarrow \boxed{A = \frac{1}{12c^2}}$

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So ⑩ $\Rightarrow W(x) = \frac{-x^4}{12c^2} + \frac{1}{12c^2}x$

$\Rightarrow W(x) = \frac{1}{12c^2}(x - x^4)$ ——— (11)

We solve equation (8) by separation of variable method;

Let $u(x,t) = X(x)T(t)$.

After solving, we have

$u(x,t) = \sum_n \sin(n\pi)x \{a_n \cos(n\pi c)t + b_n \cos(n\pi c)t\}$. ——— (12)

Applying initial conditions from (7), we have,

$u(x,0) = \sum_n a_n \sin(n\pi)x$ (9)

$\Rightarrow f(x) = \sum_n a_n \sin(n\pi)x = -W(x)$.

$\Rightarrow a_n = \frac{2}{1}\int_0^1 f(x)\sin(n\pi)x\,dx$.

$= 2\int_0^1 \frac{1}{12c^2}(x^4 - x)\sin(n\pi)x\,dx$: by (11)

$\Rightarrow a_n = \frac{1}{6c^2}\int_0^1 (x^4 - x)\sin(n\pi)x\,dx$.

and

$u_t(x,t)\Big|_{t=0} = \sum_n n\pi c b_n \sin(n\pi)x$. $= g(x)$ (say)

$\Rightarrow b_n(n\pi c) = \frac{2}{1}\int_0^1 g(x)\sin(n\pi)x\,dx$.

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$\Rightarrow b_n = 0 \quad \because g(x) = 0.$

Thus

$$v(x,t) = \sum_n a_n \sin(n\pi)x \cos(n\pi c)t \quad \text{——— (13)}$$

Hence by (11) and (13) the final solution is given by

$$u(x,t) = \sum_n a_n \sin(n\pi)x \cos(n\pi c)t + \frac{1}{6c^2}(x - x^4)$$

where $a_n = \frac{1}{6c^2}\int_0^1 (x^4 - x)\sin(n\pi)x\,dx.$

Question.

Solve the inhomogeneous I.V/B.V problem.

$u_{tt} = c^2 u_{xx} + Ax, \quad 0 < x < 1.$

$u(x,0) = 0, \quad u_t(x,0) = 0.$

$u(0,t) = 0, \quad u(1,t) = 1.$

Solution.

Given that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + Ax; \quad 0 < x < 1 \quad \text{——— (1)}$$

$$u(0,t) = 0, \quad u(1,t) = 1 \quad \text{——— (2)}$$

and $u(x,0) = 0, \quad u_t(x,0) = 0 \quad$ ——— (3)

Suppose $u(x,t) = v(x,t) + w(x)$ is the Solution of given PDE.

Then

$$(1) \Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} + \cancel{\frac{\partial^2 w(x)}{\partial t^2}}^{0} = c^2\left(\frac{\partial^2 v(x,t)}{\partial x^2} + \frac{\partial^2 w(x)}{\partial x^2}\right) + Ax$$

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$$\Rightarrow \frac{\partial^2 v(x,t)}{\partial t^2} = c^2 \frac{\partial^2 v(x,t)}{\partial x^2} + c^2 \frac{d^2 w(x)}{dx^2} + Ax \quad \text{---} (5)$$

$(2) \Rightarrow v(0,t) + w(0) = 0 \;,\; v(1,t) + w(1) = 1$

$\Rightarrow v(0,t) = 0 \;,\; v(1,t) = 0$
$w(0) = 0 \;,\; w(1) = 1$] ---- (6)

and

$(3) \Rightarrow v(x,0) + w(x) = 0 \;,\; v_t(x,0) + w_t(x,0) = 0$ ($w_t \to 0$)

$\Rightarrow v(x,0) = -w(x) \;,\; v_t(x,0) = 0$ ---- (7)

The DE. (5), with B.Cs and I.Cs in (6) and (7), is equivalent to the following two problems

$$\frac{\partial^2 v(x,t)}{\partial t^2} = c^2 \frac{\partial^2 v(x,t)}{\partial x^2} \quad \text{---} (8) \;;\quad c^2 \frac{d^2 w(x)}{dx^2} + Ax = 0 \quad \text{---} (9)$$

same as last examp $\qquad w(0) = 0,\; w(1) = 1.$

$$(9) \Rightarrow \frac{d^2 w(x)}{dx^2} = -\frac{Ax}{c^2}$$

$$\Rightarrow \frac{dw(x)}{dx} = -\frac{Ax^2}{2c^2} + B$$

$$\Rightarrow w(x) = -\frac{Ax^3}{6c^2} + Bx + C \quad \text{---} (10)$$

$$w(0) = C \Rightarrow 0 \Rightarrow \boxed{C = 0}$$

$$w(1) = -\frac{A}{6c^2} + B = 1$$

where A is given constant

$$\Rightarrow B = 1 + A/6c^2$$

$$(10) \Rightarrow w(x) = -\frac{Ax^3}{6c^2} + \left(1 + \frac{A}{6c^2}\right)x + 0$$

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$$\Rightarrow W(x) = \frac{A}{6c^2}(x - x^3) + x \quad \text{——— (11)}$$

(By last example)

$$a_n = \frac{1}{3c^2}\int_0^1 \{A(x-x^3) + 6c^2 x\} \sin(n\pi x)\, dx.$$

and $b_n = 0$

Hence the final solution is

$$u(x,t) = \sum_n a_n \sin(n\pi x)\cos(n\pi c)t + \frac{A}{6c^2}(x - x^3) + x$$

where $a_n = \frac{1}{3c^2}\int_0^1 \{A(x-x^3) + 6c^2 x\} \sin(n\pi x)\, dx.$

D' Alembert's Solution of the wave equation.

One dimensional wave equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——— (1)}$$

Let $w = x - ct$, $z = x + ct$, so that the function $u(x,t)$ is now a function of the new variables w and z.

Using chain rule for partial diffs, we have,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \quad \text{——— (2)}$$

Now $\frac{\partial w}{\partial x} = 1 = \frac{\partial z}{\partial x}$

So (2) $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}$

$$\Rightarrow \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \quad \text{——— (3)}$$

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$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \quad \longleftarrow (4)$$

Now $\frac{\partial w}{\partial t} = -c \quad ; \quad \frac{\partial z}{\partial t} = c$

So (4) $\Rightarrow \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial w} + c\frac{\partial u}{\partial z}$

$$\Rightarrow \frac{\partial}{\partial t} = -c\frac{\partial}{\partial w} + c\frac{\partial}{\partial z} \quad \longrightarrow (5)$$

Now $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)\frac{\partial u}{\partial x}$ using (3)

$$= \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial w} + \frac{\partial u}{\partial z}\right)$$

$$= \frac{\partial^2 u}{\partial w^2} + \frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z \partial w} + \frac{\partial^2 u}{\partial z^2}$$

we assume that $\frac{\partial^2 u}{\partial w \partial z} = \frac{\partial^2 u}{\partial z \partial w}$

So $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} \quad \longrightarrow (6)$

and $\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \left(-c\frac{\partial}{\partial w} + c\frac{\partial}{\partial z}\right)\left(-c\frac{\partial u}{\partial w} + c\frac{\partial u}{\partial z}\right)$

$$= c^2\frac{\partial^2 u}{\partial w^2} - c^2\frac{\partial^2 u}{\partial w \partial z} - c^2\frac{\partial^2 u}{\partial z \partial w} + c^2\frac{\partial^2 u}{\partial z^2}$$

we assume that $\frac{\partial^2 u}{\partial w \partial z} = \frac{\partial^2 u}{\partial z \partial w}$

Then $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial w^2} - 2c^2\frac{\partial^2 u}{\partial w \partial z} + c^2\frac{\partial^2 u}{\partial z^2} \quad \longrightarrow (7)$

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Substituting ⑥ and ⑦ in ①, we get,

$$\frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2}$$

$$\Rightarrow 4\frac{\partial^2 u}{\partial w \partial z} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial w \partial z} = 0$$

$$\Rightarrow \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial z}\right) = 0$$

integrating partially w.r.t 'w' we get,

$$\Rightarrow \frac{\partial u}{\partial z} = f(z)$$

$$\Rightarrow u = \int f(z)\,dz + \psi(w) \quad \text{(say)}$$

$$\int f(z)\,dz = \varphi(z)$$

or

$$u(x,t) = \varphi(z) + \psi(w)$$

$$\Rightarrow u(x,t) = \varphi(x+ct) + \psi(x-ct) \quad \text{——— (8)}$$

The solution in equation (8) is called D'Alembert solution of the wave equation.

Example. 2017,

Use D' Alembert solution of the one-D wave equation to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$$u(0,t) = 0\,; \quad u(a,t) = 0$$

$$u(x,0) = f(x)\,; \quad u_t(x,0) = g(x).$$

Solution

Given that $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ ——— ①

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$u(0,t) = 0;\ u(a,t) = 0$ ——— (2)

$u(x,0) = f(x);\ u_t(x,0) = g(x)$ ——— (3)

D'Alembert's general solution of DE (1) is given as

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

Diff. the above solution w.r.t. t, we get,

$u_t(x,t) = \phi'(x+ct)\,c + \psi'(x-ct)(-c)$

$\Rightarrow u_t(x,0) = c\phi'(x) - c\psi'(x)$

$\Rightarrow g(x) = c(\phi'(x) - \psi'(x))$

$\Rightarrow \phi'(x) - \psi'(x) = \frac{1}{c} g(x)$ ——— (4)

and $u(x,0) = \phi(x) + \psi(x)$

$\Rightarrow f(x) = \phi(x) + \psi(x)$ ——— (5)

Integrating (4), we get

$\phi(x) - \psi(x) = \frac{1}{c}\int g(x)\,dx + A$ ——— (6)

where A is constant of integration.

Adding (5) and (6), we get

$2\phi(x) = f(x) + \frac{1}{c}\int g(x)\,dx + A$

$\Rightarrow \phi(x) = \frac{1}{2}\left\{f(x) + \frac{1}{c}\int g(x)\,dx + A\right\}$

$\Rightarrow \phi(x) = \frac{1}{2}\{f(x) + G(x) + A\}$ ——— (7)

where $G(x) = \frac{1}{c}\int g(x)\,dx$ (say).

Subtracting (5) and (6), we get,

$2\psi(x) = f(x) - \frac{1}{c}\int g(x)\,dx - A$

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$$\Rightarrow \psi(x) = \frac{1}{2}\{f(x) - G(x) - A\} \quad \text{—— (8)}$$

where $G(x) = \frac{1}{c}\int g(x)\,dx$.

Here the functions $f(x)$ and $g(x)$ are defined over the interval $(0, a)$. The general D'Alembert's solution expresses $u(x,t)$ in terms of the values of the arguments at $x \pm ct$. The functions $f(x)$ and $g(x)$ may not be defined at $x \pm ct$. Therefore, before using (7) and (8) we must replace $f(x)$ and $g(x)$ by their periodic extensions $\bar{f}(x)$ and $\bar{g}(x)$.

So $u(x,t) = \varphi(x+ct) + \psi(x-ct)$

$$= \frac{1}{2}\{\bar{f}(x+ct) + \bar{G}(x+ct) + A\}$$

$$+ \frac{1}{2}\{\bar{f}(x-ct) - \bar{G}(x-ct) - A\}.$$

$$\Rightarrow u(x,t) = \frac{1}{2}\{\bar{f}(x+ct) + \bar{f}(x-ct) + \bar{G}(x+ct) + \bar{G}(x-ct)\}$$

$A - A = 0$

using boundary conditions by eq (2), we have,

$$u(a,t) = \frac{1}{2}\{\bar{f}(ct) + \bar{f}(-ct) + \bar{G}(ct) + \bar{G}(-ct)\} = 0$$

$$\Rightarrow \bar{f}(ct) + \bar{f}(-ct) + \bar{G}(ct) + \bar{G}(-ct) = 0$$

This holds only if

$$\bar{f}(ct) + \bar{f}(-ct) = 0, \quad \bar{G}(ct) - \bar{G}(-ct) = 0.$$

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$\Rightarrow \bar{f}(-ct) = -\bar{f}(ct)$; $\bar{G}(ct) = \bar{G}(-ct)$.

This shows that $\bar{f}$ is an odd function and $\bar{G}$ is an even function.

Also $u(a,t) = \frac{1}{2}\{\bar{f}(a+ct) + \bar{f}(a-ct) + \bar{G}(a+ct) - \bar{G}(a-ct)\} = 0$

$\Rightarrow \bar{f}(a+ct) + \bar{f}(a-ct) + \bar{G}(a+ct) - \bar{G}(a-ct) = 0$

This holds only if

$\bar{f}(a+ct) + \bar{f}(a-ct) = 0$; $\bar{G}(a+ct) - \bar{G}(a-ct) = 0$.

$\Rightarrow \bar{f}(a+ct) = -\bar{f}(a-ct)$; $\bar{G}(a+ct) = \bar{G}(a-ct)$.

$\Rightarrow \bar{f}(a+ct) = \bar{f}(-a+ct)$; $\bar{G}(a+ct) = \bar{G}(-a+ct)$:

because $\bar{f}$ is odd and $\bar{G}$ is even.

This shows that $\bar{f}$ and $\bar{G}$ are periodic functions of period $2a$.

Thus

$$u(x,t) = \frac{1}{2}\{\bar{f}(x+ct) + \bar{f}(x-ct) + \bar{G}(x+ct) - \bar{G}(x-ct)\} \quad \text{——— (9)}$$

Since

$$\bar{G}(x) = \frac{1}{c}\int_0^x \bar{g}(x')\,dx.$$

Therefore, $\bar{G}(x+ct) = \frac{1}{c}\int_0^{x+ct} \bar{g}(x')\,dx'$

and $\bar{G}(x-ct) = \frac{1}{c}\int_0^{x-ct} \bar{g}(x')\,dx'$.

$$\Rightarrow \bar{G}(x+ct) - \bar{G}(x-ct) = \frac{1}{c}\int_0^{x+ct} \bar{g}(x')\,dx' - \int_0^{x-ct} \bar{g}(x')\,dx'$$

$$= \frac{1}{c}\left\{\int_{x-ct}^{0} \bar{g}(x')\,dx' + \int_0^{x+ct} \bar{g}(x')\,dx'\right\}.$$

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$$\Rightarrow \bar{G}(x+ct) - \bar{G}(x-ct) = \frac{1}{c}\int_{x-ct}^{x+ct} \bar{g}(x')\,dx'.$$

Thus equation (9) becomes

$$u(x,t) = \frac{1}{2}\left\{\bar{f}(x+ct) + \bar{f}(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \bar{g}(x')\,dx'\right\}.$$

which is required solution of the given problem.

Two dimensional problems.

Example 1.

Solve the following problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K}\frac{\partial u}{\partial t}, \quad 0 \le x \le a, \; 0 \le y \le b.$$

$$u(0,y,t) = 0; \quad u(a,y,t) = 0$$

$$u(x,0,t) = 0; \quad u(x,b,t) = 0$$

$$u(x,y,0) = f(x,y).$$

Solution.

Given that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ ——— (1)

$$\left.\begin{aligned} u(0,y,t) &= 0; \quad u(a,y,t) = 0 \\ u(x,0,t) &= 0; \quad u(x,b,t) = 0 \end{aligned}\right\} \text{——— (2)}$$

$$u(x,y,0) = f(x,y) \text{ ——— (3).}$$

Suppose $u(x,y,t) = X(x)Y(y)T(t)$ is solution of the given problem.

Then

$$(1) \Rightarrow YT\frac{d^2X}{dx^2} + XT\frac{d^2Y}{dy^2} = \frac{1}{K}XY\frac{dT}{dt}.$$

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Dividing by XYT on both sides, we get,

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{KT}\frac{dT}{dt} \quad \text{---- (4)}$$

Since the L.H.S of (4) is function of x and y only and the R.H.S is function of t only, both sides are equal only if they are equal to some constant say $-\lambda^2$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda^2 \quad \text{---- (5)}$$

and $$\frac{1}{KT}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \frac{dT}{dt} = -\lambda^2 KT$$

$$\Rightarrow \frac{1}{T}T' = -\lambda^2 K.$$

Integrating, we have.

$$\ln T = -\lambda^2 kt + \ln C_1 \qquad -\lambda^2 kt.$$

$$\Rightarrow \ln T = \ln e^{-\lambda^2 kt} + \ln C_1 = \ln C_1 e^{-\lambda^2 kt}$$

$$\Rightarrow T(t) = C_1 e^{-\lambda^2 kt} \quad \text{---- (6)}$$

$$(5) \Rightarrow \frac{1}{X}X'' + \frac{1}{Y}Y'' = -\lambda^2$$

Since the L.H.S of above equation is function x and y i.e first term is a function of x only and 2nd term is function of y only if these two terms on L.H.S ar equal to

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some constants, say $-\mu^2, -\nu^2$.

$$\Rightarrow \frac{1}{X}X'' = -\mu^2 \quad ; \quad \frac{1}{Y}Y'' = -\nu^2$$

such that $-\lambda^2 = -\mu^2 - \nu^2$.

$$\Rightarrow X'' + \mu^2 X = 0 \quad ; \quad Y'' + \nu^2 Y = 0.$$

After solving these two equations, we get

$$X(x) = C_2 \cos \mu x + C_3 \sin \mu x.$$

$$Y(y) = C_4 \cos \nu y + C_5 \sin \nu y$$

Applying boundary conditions, we get,

$$u(0, y, t) = X(0) Y(y) T(t) = 0.$$

$$\Rightarrow X(0) = 0.$$

$$X(0) = C_2 = 0 \Rightarrow \boxed{C_2 = 0}$$

$$u(a, y, t) = X(a) Y(y) T(t) = 0$$

$$\Rightarrow X(a) = 0$$

$$\Rightarrow X(a) = C_3 \sin \mu a = 0.$$

For non trivial solution we choose $C_3 \neq 0$.

$$\Rightarrow \sin \mu a = 0$$

$$\Rightarrow \mu a = m\pi \quad ; \quad m = 0, \pm 1, \pm 2, \cdots$$

$$\Rightarrow \mu_m = \frac{m\pi}{a} \quad ; \quad m = 1, 2, 3, \cdots$$

$$\Rightarrow X_m(x) = C_m \sin \mu_m x = C_m \sin\left(\frac{m\pi}{a}\right)x$$

and

$$u(x, 0, t) = X(x) Y(0) T(t) = 0$$

$$\Rightarrow Y(0) = 0$$

$$u(x, b, t) = X(x) Y(b) T(t) = 0$$

$$\Rightarrow Y(b) = 0.$$

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$\Rightarrow Y_n(y) = C_n \sin \mu_n y = C_n \sin\left(\frac{n\pi}{b}\right) y$.

Thus the solution is

$U_{mn}(x,y,t) = X_m(x)\, Y_n(x)\, T_{mn}(t)$

$= C_m \sin\left(\frac{m\pi}{a}\right)x \, C_n \sin\left(\frac{n\pi}{b}\right)y \, C_{mn}\, e^{-\lambda_{mn}^2 kt}$

$\Rightarrow U_{mn}(x,y,t) = C^*_{mn} \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \, e^{-\lambda_{mn}^2 kt}$ ——— (7)

where $-\lambda_{mn}^2 = -\mu_m^2 - \nu_n^2$.

By the principle of superposition, linear combination of above solution given in (7) is also a solution.

$\Rightarrow U_{mn}(x,y,t) = \sum_m \sum_n d_{mn} C^*_{mn} \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \, e^{-\lambda_{mn}^2 kt}$

$\Rightarrow U(x,y,t) = \sum_m \sum_n C'_{mn} \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \, e^{-\lambda_{mn}^2 kt}$ ——— (8).

Applying initial condition, we have,

$U(x,y,0) = \sum_m \sum_n C'_{mn} \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y$

by (3): $f(x,y) = \sum_m \sum_n C'_{mn} \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y$

which is a double Fourier sin series

So $C'_{mn} = \frac{2}{a}\frac{2}{b}\int_0^b \int_0^a f(x,y) \sin\left(\frac{m\pi}{a}\right)x \sin\left(\frac{n\pi}{b}\right)y \, dy\, dx$ ——— (9)

Hence the final solution is given by (8) and C'_{mn} is given by (9).

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Example 2:

Solve the boundary value problem

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \quad 0 \le x \le a, \quad 0 \le y < \infty.$$

$$u(x,0) = f_1(x) \quad , \quad u(x,b) = f_2(x) \quad , \quad 0 < x < a.$$

$$u(0,y) = g_1(y) \quad , \quad u(a,y) = g_2(y) \quad , \quad 0 < y < \infty.$$

with the requirement that $u(x,y) \to \infty$ as $y \to \infty$.

Solution.

Let $u = u_1 + u_2$ be the solution of given problem. Then the given problem can be split into the following two problems.

$$\nabla^2 u_1 = 0 \quad , \quad 0 \le a \le a \quad , \quad 0 \le y \le b.$$

$$u_1(x,0) = f_1(x), \quad u_1(x,b) = f_2(x), \quad 0 < x < a$$

$$u_1(0,y) = 0 \quad , \quad u_1(a,y) = 0 \quad , \quad 0 < y < \infty.$$

and

$$\nabla^2 u_2 = 0$$

$$u_2(x,0) = 0, \quad u_2(x,b) = 0, \quad 0 < x < a$$

$$u_2(0,y) = g_1(y), \quad u_2(a,y) = g_2(y) \quad , \quad 0 < y < \infty.$$

To solve the first problem we let $u_1 = X(x)Y(y)$ be the solution.

Then

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\Rightarrow Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

$$\Rightarrow YX'' + XY'' = 0$$

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$$\Rightarrow Y X'' = -X Y''$$

$$\Rightarrow \frac{1}{X} X'' = \frac{-1}{Y} Y'' \quad \text{---} \quad (1)$$

Since the L.H.S. of (1) is function of x only and R.H.S is function of y only, both sides are equal only if they are equal to some constant say $-\lambda^2$.

$$\frac{1}{X} X'' = -\lambda^2 \quad , \quad \frac{-1}{Y} Y'' = -\lambda^2$$

$$\Rightarrow X'' + \lambda^2 X = 0 \text{ ---} (2) \qquad Y'' - \lambda^2 Y = 0 \text{ ---} (3)$$

The solution of (2) is

$$X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x$$

Applying boundary conditions, we get

$$X_n(x) = C_{2n} \sin\lambda_n x \qquad \because C_1 = 0.$$

and $\lambda_n = \frac{n\pi}{a}$, $n = 1, 2, 3, \cdots$

Given that $u_1 \to 0$ as $y \to \infty$.

So $Y_n \to 0$ as $y \to \infty$.

Solving (3), we get,

$$Y = C_3 e^{\lambda y} + C_4 e^{-\lambda y}$$

Applying boundary conditions, we get

$$Y_n(x) = C_n' e^{\lambda_n y} + C_n^* e^{-\lambda_n y}$$

$Y_n \to 0$ as $y \to \infty$, this is possible if $C_n' = 0$ So

$$Y_n(x) = C_n^* e^{-\lambda_n y}.$$

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Thus $u_{1n} = X_n(x)\,Y_n(x)$

$= C_{2n} \sin\lambda_n x \; C_n^* e^{-\lambda_n y}$

$u_n = p_n e^{-\lambda_n y} \sin\lambda_n x$ ——— (4)

where $p_n = C_{2n} C_n^*$ and $\lambda_n = \frac{n\pi}{a}$,

Now we solve the second problem as follows.

Let $u_2 = X(x)\,Y(y)$ be solution of the second problem then we can write

$Y'' + \mu^2 Y = 0$ ——— (5), $X'' - \mu^2 X = 0$ ——— (6)

solving (5) and (6), we get

(6) $\Rightarrow X_m(x) = C_m^{**} e^{\mu_m x} + C_m^{***} e^{-\mu_m x}$

and (5) $\Rightarrow Y_m(x) = C_m^{*} \sin\mu_m y$; when $\mu_m = \frac{m\pi}{b}$.

$\Rightarrow u_{2n}(x) = (C_m^* e^{\mu_m x} + C_m e^{-\mu_m x}) \sin\mu_m y$.

So $u = u_1 + u_2$

$\Rightarrow u = \sum_n p_{mn}^* \left[e^{-\lambda_n y} (e^{\mu_m x} + e^{-\mu_m x}) \right] \sin\lambda_n x \sin\mu_m y$.

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Question: A/2017

Show that the normality condition of Legendre polynomials w.r.t. weight function $w(x)=1$ over the interval $[-1,1]$ is $\int_{-1}^{1}[P_n(x)]^2\,dx=\frac{2}{2n+1}$

Solution.

The generating function for Legendre polynomials $P_n(x)$ is

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-1/2}$$

and $$\sum_{m=0}^{\infty} t^m P_m(x) = (1-2xt+t^2)^{-1/2}$$

$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m} P_n(x) P_m(x) = (1-2xt+t^2)^{-1}$$

Taking integral from $\int_{-1}^{1}$ as

$$\int_{-1}^{1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t^{n+m} P_n(x)P_m(x)\,dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2}$$

when $m=n$ then

$$\int_{-1}^{1}\sum_{n=0}^{\infty}[P_n(x)]^2 t^{2n}\,dx = \int_{-1}^{1}\frac{dx}{1-2xt+t^2} \quad \text{---(1)}$$

Now Consider

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = -\frac{1}{2t}\int_{-1}^{1}\frac{-2t}{1-2xt+t^2}\,dx$$

$$= -\frac{1}{2t}\ln|1-2xt+t^2|_{-1}^{1}$$

$$= -\frac{1}{2t}\ln|1-2t+t^2| + \frac{1}{2t}\ln|1+2t+t^2|$$

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$$= -\frac{1}{2t}\ln|1-t|^2 + \frac{1}{2t}\ln|1+t|^2$$

$$= -\frac{1}{t}\ln|1-t| + \frac{1}{t}\ln|1+t|.$$

$$= +\frac{1}{t}\ln\left|\frac{1+t}{1-t}\right| = \frac{1}{t}\ln|1+t| - \frac{1}{t}\ln|1-t|$$

$$= \frac{1}{t}\left\{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right\} - \frac{1}{t}\left\{-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \cdots\right\}.$$

$$= \frac{1}{t}\left\{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots\right\}$$

$$= \frac{1}{t}\left\{2t + \frac{2t^3}{3} + \frac{2t^5}{5} + \cdots\right\}.$$

$$= \frac{2t}{t}\left\{1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots\right\}.$$

$$= 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}$$

Now, equation ① becomes

$$\sum_{n=0}^{\infty} \int_{-1}^{1} [P_n(x)]^2 t^{2n}\, dx = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n}$$

$$\Rightarrow \int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$$ as required.

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Question. S/2018.

Define transient temperature distribution. Discuss the uniqueness of the solution of heat flow problem.

Solution.

If $u(x,t) - V(x) = w(x,t)$

then $w(x,t)$ is called transient distribution and $V(x)$ is called steady state temperature distribution.

Uniqueness of solution.

Let us consides the heat conduction problem in a very general form. So we take the heat equation as

$$u_t - K u_{xx} = \frac{1}{c\rho} s \quad ; \quad x \in (x_1, x_2); t > t_0$$

where s is the source term subject to the boundary and initial conditions

$$u(x_1, t) = \alpha(t) \qquad t > t_0$$
$$u(x_2, t) = \beta(t)$$
$$u(x, t_0) = f(x) \quad ; \quad x \in (x_1, x_2)$$

Suppose on contrary that, there are two solutions of this problem. Let us denote these solutions by u and V.

If $w = u - V$ then w must

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satisfy the equations

$$w_t - k w_{xx} = 0 \; ; \; x_1 < x < x_2 \quad \text{——— (1)}$$

$$w(x_1, t) = 0 \quad \text{——— (2)}$$

$$w(x_2, t) = 0$$

$$w(x, t_0) = 0 \quad \text{——— (3)}$$

Now we will show that $w(x,t) = 0$ so that $v(x,t) = u(x,t)$.

Let

$$I(t) = \int_{x_1}^{x_2} w^2(x,t)\, dx \quad \text{——— (4)}$$

Then clearly $I(t) \geq 0$ and $I(t_0) = 0$.

Diff. (4), w.r.t. t, we get

$$\frac{dI}{dt} = \int_{x_1}^{x_2} \frac{\partial}{\partial t} w^2(x,t)\, dx$$

$$= \int_{x_1}^{x_2} 2w\, w_t(x,t)\, dx$$

$$= 2\int_{x_1}^{x_2} w\, w_t(x,t)\, dx$$

$$= 2\int_{x_1}^{x_2} w\,(k w_{xx})\, dx \quad \text{using (1)}$$

$$= 2k \int_{x_1}^{x_2} w\, w_{xx}\, dx$$

Integrating by parts, we get

$$= 2k \left[w\, w_x \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} w_x \cdot w_x\, dx \right]$$

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by (2)

$$= 2k\left[\cancel{w(x_2,t)w_x}^{0} - \cancel{w(x_1,t)w_x}^{0} - \int_{x_1}^{x_2} w_x^2\,dx\right]$$

$$= -2k\int_{x_1}^{x_2} w_x^2\,dx \leq 0.$$

This shows that the function $I(t)$ is a decreasing or constant function. This result together with the facts that $I(t) \geq 0$ for $t \geq t_0$ and $I(t_0) = 0$ implies that $I(t) = 0$ for all $t \geq t_0$. Therefore, from eq (4) it follows that $w(x,t) = 0$.

Hence $v(x,t) = u(x,t)$.

$\Rightarrow$ solution is unique.

Question.

Discuss the uniqueness of the solution for vibrating string problem.

Solution.

Let us consider the vibrating string problem in a very general form. So we take the wave equation as

$$u_{tt} = \frac{T}{\rho} u_{xx} + \frac{1}{\rho} F(x,t)$$

subject to the boundary and initial conditions

$$u(x_1, t) = 0$$

$$u(x_2, t) = 0$$

and

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$u(x, t_0) = f(x)$

$u_t(x, t_0) = g(x)$

On Contrary suppose that there are two solutions of this problem. Let us denote these solutions by u and v. If $w = u - v$ then w must satisfy the equations

$$w_{tt} = \frac{T}{\rho} w_{xx} \quad \text{——— (1)}$$

$$w(x_1, t) = 0 \quad \text{——— (2)}$$

$$w(x_2, t) = 0$$

and $w(x, t_0) = 0 \quad \text{——— (3)}$

$$w_t(x, t_0) = 0$$

Now we will show that

$w(x, t) = 0$ so that $u(x, t) = v(x, t)$.

Let

$$I(t) = \frac{1}{2} \int_{x_1}^{x_2} (T w_x^2 + \rho w_t^2)\, dx \quad \text{——— (4)}$$

which gives the total energy of the vibrating string at time t.

Since $w(x, t)$ has continuous derivatives upto the second order in x and t, we have from (4)

$$\frac{dI(t)}{dt} = \frac{1}{2} \int_{x_1}^{x_2} \frac{\partial}{\partial t} (T w_x^2 + \rho w_t^2)\, dx$$

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$$= \frac{1}{2}\int_{x_1}^{x_2} (2\bar{I} w_x w_{xt} + 2\rho w_t w_{tt})\, dx$$

$$= \int_{x_1}^{x_2} (\bar{I} w_x w_{xt} + \rho w_t w_{tt})\, dx$$

$$= \int_{x_1}^{x_2} \bar{I} w_x w_{xt}\, dx + \int_{x_1}^{x_2} \rho w_t w_{tt}\, dx$$

$$= I_1 + I_2 \qquad \text{——— (5)}$$

where

$$I_1 = \int_{x_1}^{x_2} \bar{I} w_x w_{xt}\, dx$$

Integrating by parts, we get

$$I_1 = \bar{I} w_x w_t \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} w_t \bar{I} w_{xx}\, dx$$

$$= \bar{I}\left[w_x(x_2, t) w_t - w_x(x_1, t) w_t \right] \quad \text{by (2)}$$

($w_x(x_2, t)$ and $w_x(x_1, t)$ are 0)

$$- \int_{x_1}^{x_2} w_t \bar{I} w_{xx}\, dx$$

$$I_1 = -\int_{x_1}^{x_2} \bar{I} w_t w_{xx}\, dx$$

Now eq (5) becomes

$$\frac{dI}{dt} = -\int_{x_1}^{x_2} \bar{I} w_t w_{xx}\, dx + \int_{x_1}^{x_2} \rho w_t w_{tt}\, dx$$

$$= \int_{x_1}^{x_2} (\rho w_t w_{tt} - \bar{I} w_t w_{xx})\, dx$$

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$$\Rightarrow \frac{dI}{dt} = \int_{x_1}^{x_2} (\rho w_{tt} - \bar{\tau} w_{xx}) w_t \, dx \quad \text{---} (6)$$

From (1), we have

$$\rho w_{tt} = \bar{\tau} w_{xx}$$

$$\Rightarrow \rho w_{tt} - \bar{\tau} w_{xx} = 0$$

$$(6) \Rightarrow \frac{dI}{dt} = 0$$

$$\Rightarrow I = \text{Constant} = C$$

Also $I(t_0) = C$.

To evaluate C, we have

$$I(t_0) = \frac{1}{2} \int_{x_1}^{x_2} (\bar{\tau} w_x^2(x, t_0) + \rho w_t^2(x, t_0)) \, dx$$

From (3), we have

$$w_x(x, t_0) = 0 = w_t(x, t_0)$$

Therefore,

$$I(t_0) = 0 \qquad \forall\, t \geq t_0.$$

$$\Rightarrow \boxed{C = 0}$$

which is possible if

$$w_x = 0, \; w_t = 0 \qquad \forall\, t \geq t_0.$$

To satisfy both these conditions we must have

$$w(x, t) = \text{Constant}.$$

and the condition $w(x, t_0) = 0$ requires that $w(x, t) = 0$

Hence, $u(x, t) = v(x, t)$

$\Rightarrow$ Solution is unique.

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Question: S1 2018

Show that generating function of Legendre polynomial $P_n(x)$ is given by $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$.

Solution.

$$L.H.S = (1-2xt+t^2)^{-1/2}$$

$$= [1-t(2x-t)]^{-1/2}$$ by binomial expansion.

$$= 1 + \frac{t}{2}(2x-t) + \frac{(-1/2)(-\frac{1}{2}-1)}{2!}(-t)^2(2x-t)^2$$

$$+ \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!}(-t)^3(2x-t)^3$$

$$+ \cdots + \frac{(-1/2-1)(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(n-1))}{n!}(-t)^n(2x-t)^n$$

$$= 1 + \frac{1\cdot t(2x-t)}{2\cdot 1!} + \frac{1\cdot 3\, t^2(2x-t)^2}{2^2\, 2!} + \frac{1\cdot 3\cdot 5\, t^3(2x-t)^3}{2^3\cdot 3!}$$

$$+ \cdots + \frac{1\cdot 3\cdot 5\cdots\{2(n-1)-1\}}{2^{n-1}(n-1)!}t^{n-1}(2x-t)^{n-1}$$

$$+ \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}t^n(2x-t)^n$$

Now the coefficients of t^n in R.H.S of above becomes

$$\left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\right)(2x)^n - \left(\frac{1\cdot 3\cdot 5\cdots\{2(n-1)-1\}}{2^{n-1}(n-1)!}\right)(n-1)(2x)^{n-2}$$

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$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}x^n - \frac{1\cdot3\cdot5\cdots[2(n-1)-1]}{2\cdot(n-1)!}(n-1)x^{n-2}$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}x^n - \frac{1\cdot3\cdot5\cdots(2n-1)n(n-1)}{2\cdot(n-1)!\,n(2n-1)}x^{n-2}+\cdots$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}x^n - \frac{1\cdot3\cdot5\cdots(2n-1)n(n-1)}{n!\,(2n-1)}x^{n-2}+\cdots$$

$$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}+\cdots\right] \quad \text{---------- (1)}$$

Now consider

$$P_n(x) = \sum_{m=0}^{N} \frac{(-1)^m (2n-2m)!}{m!\,2^n (n-2m)!\,(n-m)!}x^{n-2m}$$

Put $m=0$, the co-efficient of x^n is obtained as $\frac{(2n)!}{2^n (n!)^2}$ and the Co-efficient of x^{n-2} is obtained as $\frac{(-1)(2n-2)!}{2^n (n-2)!\,(n-1)!}$.

Here $\frac{1\cdot3\cdot5\cdots(2n-1)}{n!} = \frac{(2n)!}{2^n (n!)^2}$

Hence from (1), we have.

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\,t^n.$$

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